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Optimal Investment and Consumption Strategies for a Class of Utility Functions

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Business Administration

by

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ABSTRACT OF THE DISSERTATION

Optimal Investment and Consumption Strategies
For a Class of Utility Functions

by

Nils Hemming Hakansson

Doctor of Philosophy in Business Administration
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Professor George W. Brown, Chairman

This research formalizes Irving Fisher's model of the individual under risk, and represents at the same time a generalization of Phelps' model of personal saving (Econometrica, October 1962). The objective of the individual is postulated to be the maximization of expected utility from consumption over time where the horizon is arbitrarily distant. The individual's resources consist of an initial capital position (which may be negative) and a non-capital income stream which is known with certainty but which may possess any time-shape. The individual faces both financial opportunities (borrowing and lending) and an arbitrary number of productive investment opportunities.

The interest rate is presumed to be known and invariant over time; the case when the borrowing rate exceeds the lending rate is examined for a specialized model. The returns from the productive opportunities are assumed to be random variables, whose probability distributions may differ from period to period. The basic (Fisherian) characteristic of the approach taken is that the portfolio composition decision, the financing decision, and the consumption decision are all analyzed simultaneously in <u>one</u> model. The vehicle of analysis is discrete-time dynamic programming.

Optimal consumption and investment strategies are derived for the class of utility functions $\sum_{j=1}^{\infty} \alpha^{j-1} u(c_j)$, $0 < \alpha < 1$, where c_j is the amount of consumption in period j, such that $u(c) = c^{\gamma}$, $0 < \gamma < 1$, $u(c) = -c^{-\gamma}$, $\gamma > 0$, $u(c) = \log c$, or $u(c) = -e^{-\gamma c}$, $\gamma > 0$.

The optimal consumption strategies turn out to be linear and increasing in wealth and in the present value of the non-capital income stream. In three of the four models studied, the optimal consumption strategies satisfy the properties specified by the consumption hypotheses of Modigliani and Brumberg and of Friedman precisely.

The optimal lending and borrowing strategies are found to be linear in wealth. Three of the models always call for borrowing when the individual is poor while the fourth model always calls for lending when he is sufficiently rich.

The optimal investment strategies have the surprising property that the optimal mix of risky (productive) investments in each model is independent of the individual's wealth, non-capital income stream, and impatience to consume. It is conjectured that the class of utility functions examined is the only one for which this property of the optimal investment strategies holds.

The preceding result appears to have significant implications with respect to the theory of the firm. Starting with a collection of heterogeneous individuals, each of whom is bent on maximizing (his own) utility from consumption over time, it is shown that there exists a basis for the formation of firms by sub-collections of individuals, where each sub-collection in turn possesses significant heterogeneity. Each firm so formed is found to have a well-defined (unique) objective function, which may be interpreted as imputing a precise meaning to the term "profit maximization" under risk and with respect to time. Since the capital structure of the firm is found to be unimportant, an unexpected tie-in with Proposition I of Modigliani and Miller is obtained.

CHAPTER I

THE BUILDING BLOCKS

The objective of this research is to derive optimal investment and consumption strategies for individuals from alternative but fundamental starting-points, to examine and classify their properties, and to analyze their economic implications, particularly in respect to the theory of the firm. The point of view, therefore, is essentially prescriptive, placing the study in the domain of normative decision theory.

In this chapter, the various components of the economic decision problem to be studied are constructed. The objective of the individual is postulated to be the maximization of expected utility from consumption over time where the horizon is infinitely distant. The individual's resources are assumed to consist of an initial capital position (which may be negative) and a non-capital income stream which is known with certainty but which may possess any time-shape. The individual faces both financial opportunities (borrowing and lending) and an arbitrary number of productive investment opportunities. The borrowing rate may exceed the lending rate, but each interest rate is presumed to be known and invariant over time. The returns from the productive opportunities are assumed to be random variables, whose probability distributions may differ from period to period.

The components developed in Chapter I are assembled into a formal model in Chapter II, where the main results are derived. The fundamental characteristic of the approach taken is that the portfolio composition decision, the financing decision, and the consumption decision are all analyzed simultaneously. The basic model developed in this study may therefore be viewed as a formalization of Irving Fisher's model of the individual, as given in The Theory of Interest, under risk. At the same time, it represents a generalization of Phelps' model of personal saving.[1] The vehicle of analysis is discrete-time dynamic programming.

Optimal consumption and investment strategies are derived for the class of utility functions $\sum_{j=1}^{\infty} \alpha^{j-1} u(c_j)$, $0 < \alpha < 1$, where c_j is the amount of consumption in period j, such that either the risk aversion index $-u''(x)/u'(x)$ or the risk aversion index $-xu''(x)/u'(x)$ is a positive constant for all finite $x \geq 0$. It is shown that $u(x)$ belongs to this class if and only if $u(x)$ is strictly concave and satisfies one of the three "Cauchy" equations $u(x + y) = u(x)|u(y)|$, $u(xy) = u(x) + u(y)$, or $u(xy) = u(x)|u(y)|$, i. e., $u(c) = c^{\gamma}$, $0 < \gamma < 1$, $u(c) = -c^{-\gamma}$, $\gamma > 0$, $u(c) = \log c$, or $u(c) = -e^{-\gamma c}$, $\gamma > 0$.

Section 2. 6 is devoted to a discussion of the properties of the optimal consumption strategies, which turn out to be linear and increasing in wealth and in the present value of the non-capital income stream.

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[1] Edmund Phelps, "The Accumulation of Risky Capital: A Sequential Utility Analysis," Econometrica, October 1962.

In three of the four models studied, the optimal consumption strategies satisfy the properties specified by the consumption hypotheses of Modigliani and Brumberg and of Friedman precisely. The effects of changes in impatience and in risk aversion on the optimal amount to consume are found to coincide with one's expectations. However, in response to changes in the "favorableness" of the investment opportunities, the four models exhibit an exceptionally diverse pattern with respect to consumption behaviour.

Necessary and sufficient conditions for capital growth are derived in 2.7. It is found that when the one-period utility function of consumption is logarithmic, the individual will always invest the capital available after the allotment to current consumption so as to maximize the expected growth rate of capital plus the present value of the non-capital income stream.

Section 2.8 discusses the properties of the optimal lending and borrowing strategies, which are linear in wealth. Three of the models always call for borrowing when the individual is poor while the fourth model always calls for lending when he is sufficiently rich. It appears that a positive rate of interest will always exist in an economy composed of individuals obeying one of the four models as long as the combined wealth is (substantially) positive.

The optimal investment strategies have the surprising property that the optimal mix of risky (productive) investments in each model is <u>independent</u> of the individual's wealth, non-capital income stream, and impatience to consume. It is shown in 2.9 that the optimal mix depends in each case only on the probability distributions of the returns, the interest rate, and the individual's one-period utility function of consumption. It is then conjectured in 2.11 that the class of utility functions examined is the only one for which this property of the optimal investment strategies holds.

The preceding result appears to have significant implications with respect to the theory of the firm. Starting with a collection of heterogeneous individuals, each of whom is bent on maximizing (his own) utility from consumption over time, it is shown in 2.11 that there exists a basis for the formation of firms by sub-collections of individuals, where each sub-collection in turn possesses significant heterogeneity. Each firm so formed is found to have a well-defined (unique) objective function, which may be interpreted as imputing a precise meaning to the term "profit maximization" under risk and with respect to time. Since the capital structure of the firm is found to be unimportant, an unexpected tie-in with Proposition I of Modigliani and Miller is obtained.

In Chapter III, the results obtained in Chapter II are illustrated by means of examples, and some of the applications to which the model lends itself are discussed. It is noted that the model is applicable to the balanced mutual fund as well as to endowed educational and charitable organizations. In the last chapter, the relationship between the model developed in this study and other investment and consumption models is examined.

1.1 INVESTMENT VS. CONSUMPTION

Fisher defined consumption as spending for "more or less immediate enjoyment" and investing as spending of money for "more or less deferred enjoyment."[1] Turning to the more popular authors, Loeb, for example, writes that "the purpose of investment is to have funds available at a later date for spending."[2] In a different passage he states: "Any earner who earns more than he can spend is automatically an investor. It doesn't matter in the slightest whether he realizes that he is investing."[3] While no hard and fast line can be drawn between what constitutes consumption and what constitutes investment, consumption is perhaps best viewed as the exchange of present dollars for immediate or near-immediate pleasures. Investment, by the same token, may be looked upon as the expenditure of present dollars in the hope of receiving (future) dollars at some future time.[4]

We shall now show, by intuitive reasoning with respect to the investment decision, that no reasonable a priori basis exists for treating the investment and the consumption decisions of individuals non-jointly.

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1 Irving Fisher, The Theory of Interest, New York, MacMillan, 1930, reprinted, Augustus Kelley, 1965, p. 114.

2 Gerald Loeb, The Battle for Investment Survival, New York, Simon and Schuster, revised ed., 1965, p. 129.

3 Ibid., p. 9.

4 By this distinction, the ownership and occupancy of a family home is clearly both investment and consumption. The down payment, including the difference between mortgage payments plus expenses minus the rental value, if positive, constitutes an investment. The foregone rental income constitutes consumption. The return on the investment is composed of the rental value less mortgage payments and expenses, if positive, plus the final proceeds from the sale of the house.

The consumption of food, for example, might also be called an investment in that it preserves the health necessary for survival. However, we shall not take this view here.

The investment decision - characterization. The investment decision, like most problems of decision posed in a realistic way, has two fundamental characteristics: it is sequential and it is taken under risk or uncertainty. A sequential decision problem is a problem extended in time, in which the consequences thus far of decisions taken in past periods become initial conditions for present decisions. A decision problem under risk or uncertainty is one in which the model employed does not assume perfect foresight.

The investment objective. Any normative model presumes the existence and availability of an objective function. Thus, the derivation of "optimal" investment strategies, for example, is contingent upon the proper and precise specification of the maximand. In the case of the firm, there is wide disagreement as to what its objective should be, a disagreement which shows no sign of narrowing.[1] Even if one were to adopt the classical postulate of profit maximization, one immediately runs into conceptual difficulties: what does it mean to maximize profits under risk or uncertainty? And even in the case of certainty one faces the intertemporal question: when do we maximize profits?

Since all claims to the capital of the firm reside in individuals, it seems reasonable that the objective of the firm should be at least grounded in the objectives of the individual investors of equity capital.

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[1] The various objectives suggested in the literature are too numerous to be discussed here. For a taste of the different proposals, the reader is referred to Richard Ells, The Meaning of Modern Business, New York, Columbia University Press, 1960, pp. 117-21; Charles Grainger, "The Hierarchy of Objectives," Harvard Business Review, May-June 1964; and Peter Drucker, "The Objectives of a Business," The Practice of Management, New York, Harper & Row, 1954, pp. 62-65, 126-129.

If, therefore, one views the objective of the firm as derived, in some fashion, from the investment objectives of individuals, the latter become a logical starting-point in an examination of investment objectives in general.

While the object of investment activity is capital, capital per se offers nothing to the individual until it is spent. Thus, the value, or utility, of capital is determined by the enjoyment derived from the consumption it buys. In the words of Fisher:

> Money is of no use to us until it is spent. The ultimate wages are not paid in terms of money but in the enjoyment it buys. The dividend check becomes income in the ultimate sense only when we eat the food, wear the clothes, or ride in the automobile which are bought with the check.[1]

Since consumption, therefore, is the ultimate source of all pecuniary utility, we are ultimately led, in our search for optimal investment strategies, to consider individuals' utility functions for alternative consumption programs. But this is exactly what we would do if we were interested in determining his optimal consumption strategies. It is clear at this juncture that the individual's economic choice problem is two-fold: how much to invest (or alternatively, how much of one's capital to consume presently), and how to invest that capital which is not presently consumed (i. e., how to allocate it among available opportunities). While these two aspects of the problem have been examined at length separately, a joint investigation of the problem as a whole, with the exception of Fisher's classic work,

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[1] *Op. cit.*, p. 5

The Theory of Interest, and the writings of Hirshleifer,[1] still seems to be lacking. This is the more surprising since, when the problem is viewed in this light, one is hard put to find an *a priori* reason for assuming the two decisions to be independent of one another.

1.2 THE UTILITY FUNCTION

The preferences of the individual, then, which must be translated into an objective function are his preferences for alternative consumption programs. This is so, as we have seen, because only these preferences are ultimately relevant for his decisions with respect to both consumption *and* investment.

The most significant work to date on the properties of preference systems concerning alternative consumption programs is that of Koopmans,[2] later extended by Koopmans, Diamond, and Williamson.[3] On the basis of their general significance as well as their important bearing on this study, Koopmans' findings will be briefly reviewed here.

1.2.1 Koopmans' Impatience Study

Proceeding from certain basic behavior postulates concerning the preference ordering of consumption programs which extend over an infinite future, Koopmans shows in his two papers the existence of

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1 See in particular Jack Hirshleifer, "On the Theory of Optimal Investment Decision", The Journal of Political Economy, August 1958, and Jack Hirshleifer, "Investment Decision under Uncertainty: Choice-Theoretic Approaches", Quarterly Journal of Economics, November 1965.

2 Tjalling Koopmans, "Stationary Ordinal Utility and Impatience", Econometrica, April 1960.

3 Tjalling Koopmans, Peter Diamond, and Richard Williamson, Stationary Utility and Time Perspective", Econometrica, January-April, 1964.

impatience and time perspective in a broad class of such programs. The notion of impatience goes back to Böhm-Bawerk, who in The Positive Theory of Capital advanced the idea of preference for early timing of satisfaction. In Koopmans' work, impatience is essentially taken to mean that if in any given period the consumption of the commodity bundle x is preferred to that of bundle x', then the consumption in consecutive periods of x, x' is preferred to that of x', x, all other consumption being the same. The notion of time perspective will be briefly discussed later.

While formally defined in terms of a utility function, impatience is viewed as a property of the underlying preference ordering. This implies that every utility function representing the preference ordering must have the impatience property. Consequently, impatience must be expressed in terms of an ordinal utility function. An ordinal utility function is a utility function which retains its meaning under a monotonic increasing transformation, that is, if V is a utility function, so is $U = T(V)$, where T is any monotonic transformation and $T'(V) > 0$.

The postulates assert continuity, sensitivity, and stationarity of the utility function, absence of intertemporal complementarity, and the existence of a worst and a best program. Thus, the papers essentially constitute a study of the implications of a continuous and stationary ordering of infinite consumption programs.

Notation. The bundle of n commodities consumed in period j, $j = 1, 2, 3 \ldots$ is given by

$$c_j \equiv (c_{j1}, c_{j2}, \ldots, c_{jn})$$

where $c_j \geq 0$. An infinite program will be written

$$(1\text{-}1) \qquad {}_1c = (c_1, c_2, \ldots) = (c_1, {}_2c)$$

while a constant program will be denoted

$$_{con}c = (c, c, c, \ldots)$$

Statement of the postulates. P1 (Existence and continuity). There exists a utility function $U({}_1c)$, which is defined for all ${}_1c$ such that, for all j, c_j is a point of a bounded, convex subset C of the n-dimensional commodity space. The function $U({}_1c)$ has the continuity property that, if U is any of the values assumed by that function, and if U' and U'' are numbers such that $U' < U < U''$, then there exists a positive number δ such that the utility $U({}_1c')$ of every program ${}_1c'$ having a distance $d({}_1c', {}_1c) \equiv \sup_j |c_j' - c_j| < \delta$, where $|c_j' - c_j| \equiv \max_k |c_{jk}' - c_{jk}|$, from some program ${}_1c$ with utility $U({}_1c) = U$ satisfies $U' < U({}_1c') < U''$.

Calling the set $\{{}_1c \epsilon {}_1C | U({}_1c) = U\}$, where ${}_1C \equiv CxCx \ldots$ (the infinite Cartesian product of sets C), the equivalence class defined by U, the continuity property given in P1 may be termed uniform continuity on each equivalence class. In the first paper, P1 stipulated both uniform continuity on each equivalence class and unboundedness of C which severely limits the choice of functions U. Evidently Koopmans chose to sacrifice unboundedness and with it, perhaps, some realism. The distance function, or metric, also treats all periods alike, the propriety of which may be questioned. However, remembering that the presence of impatience is the phenomenon to be established, this approach certainly provides a neutral starting point.

<u>P2 (Sensitivity)</u>. There exist first-period consumption vectors c_1, c_1' and a program ${}_2c$ from the second period on, such that

$$U(c_1, {}_2c) > U(c_1', {}_2c)$$

This postulate is clearly stronger than one which simply requires that the utility function not be a constant. The object is to keep the utility function from being insensitive to all program changes which affect a given period. The choice of the first period for this purpose is arbitrary.

<u>P3 (Limited non-complementarity)</u>. For all c_1, c_1', ${}_2c$, ${}_2c'$,

(P3a) $U(c_1, {}_2c) \geq U(c_1', {}_2c)$ implies $U(c_1, {}_2c') \geq U(c_1', {}_2c')$

(P3b) $U(c_1, {}_2c) \geq U(c_1, {}_2c')$ implies $U(c_1', {}_2c) \geq U(c_1', {}_2c')$

This postulate says that the consumption of a particular bundle of commodities in one period does not affect preferences with respect to future alternatives. This, of course, is a highly questionable proposition. It would perhaps be more palatable if total expenditures on consumption were used instead as a measure of satisfaction, but this idea is rejected by Koopmans. However, here one runs into the ratchet principle.[1]

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[1] The ratchet principle essentially states that the utility of consumption in a given period is strongly conditioned on the highest level of consumption previously experienced, particularly if this level is of recent origin. This point was first made by James Duesenberry in <u>Income, Saving and the Theory of Consumer Behavior</u>, Cambridge, Massachusetts, Harvard University Press, 1949, pp. 84-85, 114-116.

As a consequence of P3, it can readily be shown that $U({}_1c)$ may be written

$$U({}_1c) = Y(u(c_1), T({}_2c))$$

where u and T are uniformly continuous on each equivalence class and Y is continuous and increasing in u and T.

<u>P4 (Stationarity).</u> For some c_1 and all ${}_2c$, ${}_2c'$

$$U(c_1, {}_2c) \geq U(c_1, {}_2c') \text{ if and only if } U({}_2c) \geq U({}_2c')$$

This postulate asserts that there exists a subset of programs that differ only from the second period on, the ordering of which is not changed by advancing the timing of each consumption vector by one period. It should be pointed out that the ordering in question applies only to the present. The passage of time is completely outside the scope of the postulate set — thus, the question of changes in preferences as a function of time is not considered.

It can be shown that P3b and P4 together imply

$$U(c_1, {}_2c) \geq U(c_1, {}_2c') \text{ if and only if } U({}_2c) \geq U({}_2c') \text{ for } \underline{\text{all}} \quad c_1, c_2, {}_2c'$$

Since Y(u, T) is increasing in T, P4 is equivalent to

$$T({}_2c) \geq T({}_2c') \text{ if and only if } U({}_2c) \geq U({}_2c')$$

Consequently, there exists a monotonic transformation H such that

$$T({}_2c) = H(U({}_2c)), \quad H'(U) > 0$$

Thus, letting $V(u, U) \equiv Y(u, H(U))$, $V(u, U)$ preserves the preference defined by $U({}_1c)$ so that we obtain the recurrence relation

$$(1\text{-}2) \qquad U({}_1c) = V(u(c_1), U({}_2c))\,.$$

where V is continuous and increasing in u and U. Clearly, the function V, which Koopmans calls the aggregator function, may be written in recursive form:

$$(1\text{-}3) \qquad U({}_1c) = V(u(c_1), V(u(c_2), V(u(c_3) \ . \ . \ .$$

Since u and U are continuous, the range of each is an interval. The question of whether these intervals include their own endpoints is settled by the last postulate in favor of inclusion.

<u>P5 (Extreme programs).</u> There exist programs ${}_1\underline{c}$ and ${}_1\bar{c}$ such that

$$U({}_1\underline{c}) \leq U({}_1c) \leq U({}_1\bar{c}) \text{ for all } {}_1c$$

By separate monotonic increasing transformations, we may cause the range of both u and U in (1-2) to coincide with the unit interval without altering the preference ordering. Clearly, this will also require a corresponding transformation of V. We then obtain directly

$$U({}_1\underline{c}) = 0, \quad U({}_1\bar{c}) = 1$$

$$u(\underline{c}_1) = 0, \quad u(\bar{c}_1) = 1$$

By monotonicity, this gives

$$V(0, 0) = 0, \quad V(1, 1) = 1$$

Consequently, the domain of V is now the unit square and the range the (closed) unit interval.

Thus, the ordinal properties of U have permitted us to derive a (non-unique) one-period utility function u by which <u>all</u> consumption vectors c_j may be evaluated, $j = 1, 2, \ldots$.

Impatience defined. At this point, we are in a position to formalize the definition of impatience. To simplify the notation, we shall let $u_j \equiv u(c_j)$, that is, u_j is the immediate utility level associated with c_j of a program.

Definition. A program ${}_1c$ with utility levels u_1, u_2 in the first two periods and utility level $U_3 \equiv U({}_3c)$ from the third period on is said to meet the strong impatience condition if $V(u_1, V(u_2, U_3)) > V(u_2, V(u_1, U_3))$ whenever $u_1 > u_2$ and the weak impatience condition if $V(u_1, V(u_2, U_3)) \geq V(u_2, V(u_1, U_3))$ whenever $u_1 > u_2$.

What this definition says is that if the first-period consumption vector c_1 is interchanged with the second-period consumption vector c_2, aggregate utility is decreased if c_1 is preferred to c_2, and conversely. This definition views impatience as a property of a program ${}_1c$; it may also be said to exist in the point (u_1, u_2, U_3) of the utility space when the defining conditions hold.

Existence of impatience. On the basis of postulates P1-P5 the following result is obtained.

Theorem 1. If P1-P5 are satisfied, a program ${}_1c$ with first and second period utilities $u_1 = u(c_1)$ and $u_2 = u(c_2)$ such that $u_1 > u_2$ and with utility $U_3 = U({}_3c)$ from the third period on meets the condition of strong impatience in each of the following three zones:

1) $U({}_{con}c_1) \leq U_3 \leq \bar{U}$ where $\bar{U}$ is the solution to $V(u_2, \bar{U}) = u_1$ if a solution exists; otherwise $\bar{U} = 1$

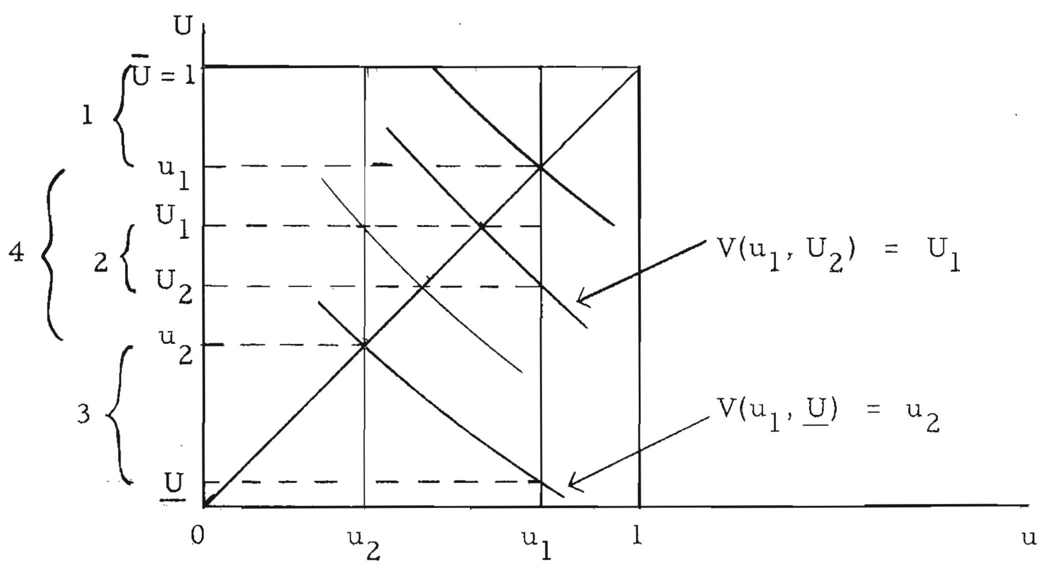
2) $U(c_2, c_1, c_2, c_1, \ldots) \leq U_3 \leq U(c_1, c_2, c_1, c_2, \ldots)$

3) $\underline{U} \leq U_3 \leq U({}_{con}c_2)$ where $\underline{U}$ is the solution to $V(u_1, \underline{U}) = u_2$ if a solution exists; otherwise $\underline{U} = 0$.

With this result, Koopmans has shown that impatience, which is usually viewed as a psychological phenomenon, is also a consequence of quite elementary properties attributed to a utility function in which the horizon is infinite. This is a significant accomplishment indeed. A geometric representation of the three impatience zones of Theorem 1 in which the scales of u and U have been equated is given in Fig. 1.

Koopmans also shows that when weak (strong) time perspective (to be discussed below) is present, one can prove that there exists weak (strong) impatience in the entire (open) interval (u_2, u_1), which of course includes zone 2 (see Fig. 1).

Concerning the outlying intervals $0 \leq U_3 < \underline{U}$ and $\bar{U} < U_3 \leq 1$, nothing conclusive can be said about impatience in them when they are non-empty. Both impatience and strong patience may exist, where strong patience is said to exist in (u_1, u_2, U_3) if



1, 2, 3, - zones of strong impatience

4 - zone of strong (weak) impatience given strong (weak) time perspective

Figure 1. Zones of Impatience

$$V(u_1, V(u_2, U_3)) < V(u_2, V(u_1, U_3))$$

whenever $u_1 > u_2$.

Time perspective. Time perspective is defined by Koopmans as follows: Let $(x_1, x_2, x_3, \ldots)$ and $(y_1, y_2, y_3, \ldots)$ be two consumption programs such that $U_1 \equiv U({}_1x) > U_2 \equiv U({}_1y)$. Now postpone each program by one period, inserting consumption vector z in the vacated first period. Then, by P4 and P3b, $U_3 \equiv U(z, x_1, x_2, \ldots) > U_4 \equiv U(z, y_1, y_2, \ldots)$. U will now be said to have the property of weak time perspective if $U_3 - U_4 \leq U_1 - U_2$, and the property of strong time perspective if $U_3 - U_4 < U_1 - U_2$. Since time perspective is defined in terms of utility differences, it is clearly not an ordinal property, i. e., a property of all utility functions representing the preference ordering. However, time perspective is imputed to the preference ordering itself when at least one utility function representing it has that property.

A cardinal utility function. Koopmans suggests that a general discount factor, $\alpha(U)$, be defined by the identity

$$(1\text{-}4) \qquad \alpha(U) \equiv \left.\frac{\partial V(u, U)}{\partial U}\right|_{u = W^{-1}(U)}$$

that is, as a function of the overall level of satisfaction achieved and provides as an example a utility function with a discount factor which decreases in U. ($U = W(u)$, denoted the correspondence function, is the solution to the equation $V(u, U) = U$.) It can be shown that (1-4) is invariant under differentiable monotonic transformations.

It should be observed that if the scales of u and U are equated and if F is an increasing transformation, the function $F^{-1}(V(F(u), F(u)))$ preserves the form $u = V(u, u)$. Thus, this form is also appropriate under risk because among the infinite number of ordinal utility functions F(u) there must surely be one, say $\hat{F}(u)$, which is cardinal, i.e., such that $E[\hat{F}^{-1}(V(\hat{F}(u), \hat{F}(u)))]$ correctly reflects the preference ordering.

An example of a utility function which satisfies postulates P1 through P5 is given by

$$
\begin{aligned}
(1\text{-}5) \qquad U({}_1c) &= u(c_1) + \alpha U({}_2c) \qquad\qquad 0 < \alpha < 1 \\
&= \sum_{j=1}^{\infty} \alpha^{j-1} u(c_j)
\end{aligned}
$$

which we recognize as the discounted sum of all future one-period utilities, a form used almost exclusively so far in economic analysis.

Differentiating (1-5) with respect to U we obtain

$$\frac{\partial V(u, U)}{\partial U} = \alpha$$

that is, the discount factor given by (1-4) is constant, which agrees with the conventional interpretation. Thus, it is clear that the utility function (1-5) implies that impatience exists in all parts of the program space.

It was noted by Koopmans that the addition of a stronger version of the non-complementarity postulate to the set P1 - P5 leaves this utility function as the only function which is consistent with the expanded postulate set. The additional postulate is given by

P3'. For all c_1, c_2, ${}_3c$, c'_1, c'_2, ${}_3c'$

(3'a) $U(c_1, c_2, {}_3c) \geq U(c'_1, c'_2, {}_3c)$ implies $U(c_1, c_2, {}_3c')$

$$\geq U(c'_1, c'_2, {}_3c')$$

(3'b) $U(c_1, c_2, {}_3c) \geq U(c'_1, c_2, {}_3c')$ implies $U(c_1, c'_2, {}_3c)$

$$\geq U(c'_1, c'_2, {}_3c')$$

This result can be demonstrated by reference to a study by Debreu, which gives the conditions under which one can find a monotonic transformation such that the function $U({}_1c)$ may be written

(1-6) $$U({}_1c) = u_1(c_1) + u_2(c_2) + U_3({}_3c)$$ [1]

Since these conditions are satisfied by P1 - P5 and P3', (1-6) is implied by the enlarged postulate set. However, writing (1-6) as a recurrence relation, i.e., in the form (1-2), by utilization of the stationarity postulate P4, we obtain as the only possibility.

(1-5) $$U({}_1c) = u(c_1) + \alpha U({}_2c) \qquad 0 < \alpha < 1$$

$$= \sum_{j=1}^{\infty} \alpha^{j-1} u(c_j)$$

Since the form (1-5) is preserved only by a positive linear transformation, P1 - P5 and P3' may be viewed as postulates which define a cardinal utility function, while the weaker set P1 - P5 defines an ordinal utility function. Thus, we need not, in the case of (1-5) at

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[1] See Gerard Debreu, "Topological Methods in Cardinal Utility Theory," Mathematical Methods in the Social Sciences, Stanford, Stanford University Press, 1960, pp. 19-25.

least, concern ourselves with how to find the particular transformations which give us a cardinal utility function from a given ordinal one. This step would, of course, always be necessary when the available consumption programs are subject to risk and only ordinal utility functions are known.

1.2.2 Properties and Limitations of the Utility Function

In this study, we shall confine our attention to utility functions of the form (1-5). The most serious drawbacks of this class of functions are undoubtedly the additivity property, which is primarily a consequence of postulates 3 and 3', and the constancy of the discount factor α. As suggested earlier, the assumptions of non-complementarity are particularly limiting when consumption is treated as a commodity vector. By focusing on total (dollar) consumption alone, certain types of complementarity between commodities need not be ruled out. Consequently, we shall choose to be concerned with the level of consumption rather than the composition of the consumption basket. Thus, the utility function will be assumed to be defined on all possible programs $(c_1, c_2, c_3, \ldots)$ where c_j, $j = 1, 2, \ldots$, is the amount of consumption in period j.

To gain a better understanding of the nature of the limitations inherent in utility functions of the form (1-5), let us examine some concrete examples. As a case in point, let us consider the function $u(c) = \log c$ and pose the problem of finding different consumption programs between which the utility function (1-5) requires the individual to be indifferent. For example, we might attempt to find the consumption programs ${}_1c'$ and ${}_1c''$ which are equivalent to the constant program ${}_1c = (10,000,\ 10,000,\ \ldots)$ and which are such that consumption increases and decreases, respectively, at the rate of 5 percent per period. Formally, we obtain from the equation $U({}_1c) = U({}_1c') = U({}_1c'')$

$$\sum_{j=1}^{\infty} \alpha^{j-1} \log(10,000) = \sum_{j=1}^{\infty} \alpha^{j-1} \log(1.05^{j-1} c_1')$$

$$= \sum_{j=1}^{\infty} \alpha^{j-1} \log(.95^{j-1} c_1'')$$

which, upon solution, gives the values shown in Table I for different rates of impatience. In appraising these indifferences, the reader should remember that their evaluation may be confounded by the assumption of an infinite horizon.

We shall postulate that $U(c_1, c_2, \ldots)$ is monotone increasing in each of its arguments which is to say that the individual always prefers more consumption to less. It will also be assumed that the individual obeys the von Neumann-Morgenstern postulates[1] when confronted with consumption prospects which are subject to risk. Since in this axiom system expected utility correctly reflects preferences, the individual will wish to behave, whenever risk is present, so as to maximize the expected utility obtainable from consumption over time. Finally, we shall assume that he is a risk averter[2] with respect to

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1 See John von Neumann and Oskar Morgenstern, Theory of Games and Economic Behavior, Princeton, Princeton University Press, 1947, pp. 26-27 and the Appendix.

2 Let ${}_1c'$ and ${}_1c''$ be two possible consumption programs. Then $U(\theta\,{}_1c' + (1-\theta)\,{}_1c'') > \theta U({}_1c') + (1-\theta)U({}_1c'')$ for all ${}_1c'$, ${}_1c''$ such that $U({}_1c') \neq U({}_1c'')$ and all θ such that $0 < \theta < 1$. That is, the individual prefers the certain program $\theta\,{}_1c' + (1-\theta)\,{}_1c''$ to the prospect of obtaining ${}_1c'$ with probability θ and ${}_1c''$ with probability $1-\theta$.

TABLE I

EQUIVALENT CONSUMPTION PROGRAMS WHEN $u(c) = \log c$

Rate of Patience α	Amount of Consumption in First Period			Number of Periods Until Consumption Levels Are Equal
	Increasing Program (5% per period)	Constant Program	Decreasing Program (5% per period)	
.50	$9,070	$10,000	$11,025	2
.67	8,630	10,000	11,575	3
.80	7,835	10,000	12,930	5
.90	6,140	10,000	16,700	10
.95	3,770	10,000	27,900	20

consumption, which implies that U is strictly concave. This assumption, which has a high degree of acceptance, is crucial to all the results which follow. But if U is monotone increasing and strictly concave, it follows trivially that $u(c_j)$ is likewise.

The utility function (1-5) is defined on infinite programs, which may seem to be out of step with the fact that man's lifespan is finite. However, we shall argue that an individual's preferences generally extend beyond his own lifetime. First, his departure point is indefinite; it therefore behooves him to be conservative in reference to his planning horizon. Second, he usually wishes to provide in some form for his heirs and successors - it is in fact this benevolence which keeps man from perishing from the earth. In the first twenty years or so, each of us depends on someone else for the economic goods he enjoys. Consequently, man has, during his lifetime, both the moral and legal right to supplement his own preferences with regard to consumption with the perceived preferences of his successors - to infinity.

This is not to say that an investigation of finite programs would be without merit. However, if this approach is used, one is faced with the problem of determining just <u>where</u> the horizon is. For the reasons given, coupled with the fact that a utility function with a distant (but finite) horizon in which impatience is present is closely approximated by the same function with the horizon extended to infinity,[1] the idea

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[1] For example, in the case of function (1-5) with $\alpha = .9$, 99.9 percent of the utility obtained form a constant consumption level is associated with the first 64 periods. More generally, if the utility of consumption amount c in a particular period is u, then the contemporary utility of c 64 periods later is .001 u.

of evaluating infinite programs appears intuitively much more satisfactory.

The preference orderings we have discussed have been considered to arise independently of the opportunities faced by the individual. This is in agreement with economic tradition, which has always separated preference from opportunity. Thus, in the ensuing models, modification of an individual's preferences in the light of experience is ruled out. An attempt to grapple with the question of allowing for flexibility of future preference has been made by Koopmans.[1]

1.2.3 Note on the Boundedness of the Utility Function

It has been shown by Arrow (who credits the discovery of the proof to Menger) that a von Neumann-Morgenstern utility function is bounded.[2] Since some of the functions u(c), and hence U, which will be employed in Chapter II are unbounded, there would appear to be cause for questioning the results obtained with these functions.

There are two bases on which the validity of using unbounded functions as utility indicators may be defended. First, as long as one never leaves that part of the domain of the (unbounded) function for which its value is finite, the unbounded part of the function might as well be "cut off." In the ensuing models, by eliminating the possibility of starting out in the trapping state (to be discussed in 2.7) in Models

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1 Tjalling Koopmans, "On Flexibility of Future Preference," Human Judgments and Optimality, ed. Maynard Shelley and Glenn Bryan, New York, John Wiley & Sons, 1964.

2 Kenneth Arrow, Bernoulli Utility Indicators for Distributions Over Arbitrary Spaces, Technical Report No. 57, Department of Economics, Stanford University, July 1958.

II and III, the results are indeed the same as they would be if only the bounded part of the function u(c) had been employed. However, if a bound were placed on u(c) in Model I, a solution would also exist, though probably not in closed form, when the convergence condition (2-29) does not hold.

A second avenue of defense would be to say that the continuity postulate, on which the boundedness of the utility indicator depends, is unnecessarily restrictive and that it should be modified (which would be easy enough) to permit the utility function to be unbounded.[1]

1.3 THE OPPORTUNITY SET

So far, we have endowed the individual with a set of preferences which falls in the general "preference class" represented by the utility function (1-5). Thus, we have in effect equipped him with the power to know what he wants. We must now, in order to complete the definition of his decision problem, specify the opportunities which his environment presents him.

1.3.1 Opportunities for Decision

We shall grant the individual free will in the sense that he has the power to make any choice in accordance with his own preferences. We shall further assume that he functions in a free economy - by this, we mean that he will not be prevented, by any non-deterministic force, from including in his choice set any opportunity offered by his environment.

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[1] A critique of the continuity postulate may be found in Duncan Luce and Howard Raiffa, Games and Decisions, New York, John Wiley, 1957, p. 27.

To simplify the exposition, we shall postulate that opportunities for decision occur at equally spaced points in time. The time between two decision points will simply be denoted the (decision) period. Thus, the individual in our model is left no choice but to decide, at each decision point, how much to set aside for consumption in the period immediately following as well as how to allocate, or reallocate, the remainder of his capital, including any borrowings he may decide upon, among the available opportunities for investment.[1]

1.3.2 Opportunities for Non-Capital Income

We shall consider individuals who have an opportunity to receive a salary or other non-capital income such as a pension, alimony, welfare payments, unemployment compensation, or the income from a trust.

The choice of non-capital income stream, to the extent that a non-trivial choice exists,[2] undoubtedly depends in part on the individual's consumption preferences. But there are also other important preferences, such as the disutility of labor, which enter when a choice among non-capital income streams is made. We shall give these

[1] In the deterministic case, it would of course be possible to make all decisions in advance.

[2] We shall say that a non-trivial choice exists only when the selection of a particular non-capital income stream affects other opportunities or is dependent on the individual taking a particular action involving disutility. An income stream which requires no action or an action involving no disutility and which affects no other opportunities would <u>always</u> be chosen as a result of the monotonicity of U (trivial choice). An example of the non-trivial case would be the selection of an income stream from a set of alternative job opportunities where the selection of any one job precludes the selection of certain other ones. The second situation is exemplified by the opportunity to choose a trust income "with no strings attached."

non-consumption considerations the benefit of the doubt and simply accept the resultant choice. The non-capital income stream will thus be viewed as exogenously determined in the model to be presented. For example, one may rationalize that the individual's choice of non-capital income stream is primarily governed by what he wants to do (which may be nothing), the question of remuneration being of secondary importance.

We shall also make the rather strong assumption that the individual's non-capital income stream, the installments of which will be received at the end of each period, is known with certainty.

The building blocks discussed so far are similar to those used by Phelps, who formulated the basic modeling structure adopted in this study.[1] However, in the next two sections, we shall extend considerably the opportunity set considered by him. In Phelp's model, all capital not currently consumed is subject to the same probability law, which is invariant over time. In the following, we shall introduce the possibility of choice among an arbitrary number of risky (productive) opportunities, which may be time-dependent, as well as the opportunity to borrow and lend.

1.3.3 Productive Investment Opportunities

We shall postulate the existence of a finite number of investment opportunities, in which, for each, the amount invested may be selected from an infinite number of possible choices. The basic premise upon which we shall base our model is that the return promised by

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[1] Phelps, *op. cit.*

each opportunity is a random variable. To paraphrase, we are stipulating that the only thing that is certain about the return from an investment is that is is uncertain. This, of course, is essentially what J. P. Morgan expressed when, asked what he thought about the stock market, he replied, "It will fluctuate."

In order to reduce the complexity of the investment situation as much as possible, while still retaining the stochastic nature of returns, we shall make the following second-order assumptions:

1. All investment opportunities are of the point-input, point-output type, i. e., investment and realization take place instantaneously rather than over time.
2. All investments are realizable in cash at the end of each period.
3. The amount invested in an opportunity may be any real number. This number is non-negative unless a short sale is made.
4. The return from each investment opportunity is proportional to the amount invested (constant returns to scale).
5. There are no non-proportional conversion costs or taxes. (By 4, proportional costs or taxes present no difficulties.)
6. We shall arbitrarily define a short sale as the opposite of a long investment. That is, if the individual sells opportunity i short in the amount θ, he will receive θ immediately (to do with as he pleases) in return for the obligation to pay the transformed value of θ at the end of the period. The case in which the seller is required to maintain a deposit to cover a short sale and, analogously, the case in which he only needs to put up the maximum amount he may lose in making a long investment, can also be handled without difficulty in the ensuing models.

While these assumed conditions undoubtedly are too restrictive to be representative of many real-world investment opportunities, they clearly hold approximately for stocks, bonds, and other liquid investment vehicles.

In accordance with the preceding, let us denote the transformation in period j of a unit of capital invested in opportunity i by β_{ij}; that is, if we invest an amount θ in i at the beginning of the period, we would obtain $\beta_{ij}\theta$ at the end of the period (β_{ij} is the random variable). We shall postulate that β_{ij} is non-negative and bounded from above for all i and j. What this means, of course, is that when one is in a long position, one can at most lose one's investment, and that a finite amount invested will always bring a finite return over a finite time period. Both of these propositions certainly seem reasonable.

We shall assume that no combination of productive investment opportunities exists which provides, with probability 1, a return at least as high as the borrowing rate of interest. (We shall, in the next section, postulate the existence of positive rates of interest such that if $r_L - 1$ is the rate at which individuals may save and $r_B - 1$ the rate at which they may borrow, then $r_B \geq r_L$.) Letting i = 1 denote the financial opportunities and i = 2, . . . , M_j the productive opportunities available in period j, this implies that the β_{ij} satisfy the inequality

$$\Pr\left\{\sum_{i=2}^{M_j} (\beta_{ij} - r_B)\,\theta_{ij} < 0\right\} > 0 \qquad \text{all } j$$

for all finite numbers $\theta_{ij} \geq 0$ such that $\theta_{ij} > 0$ for at least one i.

We shall also stipulate that no combination of short sales exists in which the probability is zero that a loss will exceed the lending rate of interest. Thus, the β_{ij} also satisfy the inequality

$$\Pr\left\{ \sum_{i \in S_j} (\beta_{ij} - r_L)\theta_{ij} < 0 \right\} > 0 \qquad \text{all } j$$

for all finite numbers $\theta_{ij} \leq 0$ such that $\theta_{ij} < 0$ for at least one i, where S_j is the set of opportunities which can be sold short in period j. In addition, we shall assume that the β_{ij} are such that

$$\Pr\left\{ \sum_{\substack{i=2 \\ i \notin S_j^*}}^{M_j} \beta_{ij}\theta_{ij} - \sum_{k \in S_j^*} \beta_{kj}\theta_{kj} < 0 \right\} > 0 \qquad \text{all } j$$

for all finite numbers $\theta_{ij} \geq 0$ and all $S_j^* \subseteq S_j$ such that $\sum_{\substack{i=2 \\ i \notin S_j^*}}^{M_j} \theta_{ij} = \sum_{k \in S_j^*} \theta_{kj}$, and $\theta_{ij} > 0$ for at least one i. The last inequality states that no combination of productive investments made from the proceeds of any short sale can guarantee against loss.

When $r_B = r_L$, the three preceding conditions reduce to

$$\Pr\left\{ \sum_{i=2}^{M_j} (\beta_{ij} - r)\theta_{ij} < 0 \right\} > 0 \qquad \text{all } j$$

for all finite θ_{ij} such that $\theta_{ij} \geq 0$ for all $i \notin S_j$ and $\theta_{ij} \neq 0$ for at least one i. We shall refer to these restrictions on the distributions of the β_{ij} as the "no-easy-money condition".

At each decision point, we shall assume that the probability distributions of return are known for at least the subsequent period. That is, the distribution functions

$$F_j(x_2, x_3, \ldots, x_{M_j}) \equiv \Pr\{\beta_{2j} \leq x_2, \beta_{3j} \leq x_3, \ldots, \beta_{M_j j} \leq x_{M_j}\}$$

will be assumed to be known at the beginning of the j^{th} period, and generally earlier. The F_j will also be assumed to be independent.

In real world situations, the individual would of course be forced to derive his own subjective probability distributions.[1] Numerous descriptions of how this may be accomplished, on the basis of postulates presupposing certain consistencies in behavior, are available in the literature; see for example the accounts of Savage[2] and Marschak.[3]

The realm of risk is generally said to prevail when the probabilities of the possible outcomes in a decision situation are known, while uncertainty is said to exist when these probabilities are unknown. By this classification, our problem, as defined so far, clearly falls in the realm of decision-making under risk. However, the distinction between the two categories is not as sharp as the preceding definition might suggest. To show this, let us for a moment consider decision-making under uncertainty.

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1 Even if he adopts what he considers to be an objective probability distribution, the very act of adoption makes the distribution subjective.

2 Leonard Savage, The Foundations of Statistics, New York, John Wiley, 1954, Ch. 3.

3 Jacob Marschak, "Decision-making," Working Paper No. 93, Western Management Science Institute, University of California at Los Angeles, December 1965, pp. 18-20.

The first observation to be made, as McKean has pointed out, is that in taking a position on an issue (under uncertainty), an individual implicity quantifies considerations which he refuses to quantify explicitly.[1] Going one step further, it follows as a theorem that a decision-maker who observes a certain measure of consistency in deciding under uncertainty in fact imputes a probability distribution over the possible outcomes, regardless of what criterion is used. This distribution is such that if it is used to solve the decision problem under risk, it will give the same solution as was obtained under uncertainty with the given criterion.[2] As a result, if one is committed to this (highly reasonable) measure of consistency, it would seem that one might as well convert the decision problem to one under risk by searching for the necessary probability distribution(s).

In conclusion, let it be said that probabilities are not something which only theoreticians consider. To quote a leading investment banker: "We are not seeking sure things - we are seeking probabilities."[3]

1.3.4 Financial Opportunities

As alluded earlier, we shall postulate that the individual may engage in both borrowing and lending operations. The interest rates for both activities will be assumed to be known with certainty and to be

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[1] Roland McKean, Economics of Defense, Santa Monica, The RAND Corporation, P-2926, July 1964, p. 12.

[2] For a detailed exposition of this result and the underlying postulates, see Duncan Luce and Howard Raiffa, Games and Decisions, New York, John Wiley, 1959, pp. 287-294.

[3] Sidney Homer, Bond Investment Policy for Pension Funds, New York, Solomon Brothers and Hutzler, 1964, p. 14.

invariant over time. We shall also stipulate that $r_B \geq r_L > 1$, where, as stated earlier, $r_L - 1$ is the lending rate and $r_B - 1$ is the borrowing rate.

While no absolute limit will be placed on the amount an individual may borrow, it will be assumed that the individual's debt must at all times be fully secured by his resources. In this connection, it should be noted that as long as the individual's debt is smaller than the present value (on the basis of the borrowing rate) of his (certain) non-capital income stream at the end of each period, he will always have the resources to pay back both interest and principal with probability 1 under our assumptions.[1] This value, then, would logically seem to be one of the induced upper limits on borrowing with which we have reason to be concerned.

[1] Presumably, no rational lender would therefore hesitate to lend up to the amount of the present value of the non-capital income stream as long as the debtor always pays his interest and consumes and invests so as to be able, with probability 1, to do so without extending his debt beyond this limit.

CHAPTER II

THE MODEL AND ITS IMPLICATIONS

In this chapter, we shall combine the building blocks developed in the previous chapter into a formal model. We shall then seek the solution to our optimization problem for certain utility functions and examine the properties and implications of the results obtained.

2.1 SUMMARY OF NOTATION

In order to have the notation developed in the previous chapter in one place, we shall summarize it, with certain obvious extensions, below:

c_j - amount of consumption in period j, where $c_j \geq 0$ (decision variable)

$U(c_1,c_2,c_3,\ldots)$ - the utility function, defined over all possible consumption programs $(c_1,c_2,c_3,\ldots)$. The class of functions to be considered is that of the form

$$U(c_1,c_2,c_3,\ldots) = u(c_1) + \alpha U(c_2,c_3,c_4,\ldots) \tag{1-5}$$

$$= \sum_{j=1}^{\infty} \alpha^{j-1} u(c_j),\ 0 < \alpha < 1$$

$u(c)$ is assumed to be monotone increasing, twice differentiable, and strictly concave for $c \geq 0$. The objective in each case is to maximize $E[U(c_1,c_2,\ldots)]$, i.e., the expected utility derived from consumption over time

x_j - amount of capital (debt) on hand at decision point j (the beginning of the jth period) (state variable)

y_j - income received from non-capital sources at the <u>end</u> of period j, where $0 \le y_j < \infty$

M_j - the number of investment opportunities available in period j

S_j - the subset of investment opportunities which it is possible to sell short in period j

z_{ij} - amount invested in opportunity i, $i = 1, \ldots, M_j$, at the beginning of the jth period (decision variable)

$r_B - 1$ - borrowing rate of interest

$r_L - 1$ - lending rate of interest, where $r_B \ge r_L > 1$

β_{ij} - transformation of capital invested in opportunity i in the jth period, per unit of capital so invested (random variable). That is, if we invest an amount θ in i at the beginning of the period, we will obtain $\beta_{ij}\theta$ at the end of that period (constant returns to scale). The joint distribution functions F_j of the β_{ij}, $i = 1, \ldots, M_j$, are assumed to be known for all j.

<u>Properties of β_{ij}:</u>

1) $0 \le \beta_{ij} < \infty$ for all i, j

2) $\beta_{1j} = \begin{cases} r_B & \text{when borrowing} \\ r_L & \text{when lending} \end{cases}$

3) $\Pr\{\sum_{i=2}^{M_j}(\beta_{ij} - r_B)\theta_{ij} < 0\} > 0$ all j and all finite $\theta_{ij} \geq 0$ such that $\theta_{ij} > 0$ for at least one i

4) $\Pr\{\sum_{i\in S_j}(\beta_{ij} - r_L)\theta_{ij} < 0\} > 0$ all j and all finite $\theta_{ij} \leq 0$ such that $\theta_{ij} < 0$ for at least one i

5) $\Pr\{\sum_{\substack{i=2 \\ i\notin S_j^*}}^{M_j}(\beta_{ij}\theta_{ij} - \sum_{k\varepsilon S_j^*}\beta_{kj}\theta_{kj} < 0\} > 0$ all j, all $S_j^* \subseteq S_j$ and all finite $\theta_{ij} \geq 0$ such that $\sum_{\substack{i=2 \\ i\notin S_j^*}}^{M_j} \theta_{ij} =$

$\sum_{k\varepsilon S_j^*}\theta_{kj}$ and $\theta_{ij} > 0$ for at least one i

When $r_B = r_L = r$, the "no-easy-money condition" 3) - 5) reduces to

5') $\Pr\{\sum_{i=2}^{M_j}(\beta_{ij} - r)\theta_{ij} < 0\} > 0$ all j, all finite θ_{ij} such that $\theta_{ij} \neq 0$ for at least one i and $\theta_{ij} \geq 0$ for all i $\notin S_j$

$f_j(x_j)$ - expected utility obtainable from consumption over all future time, evaluated at decision point j, when initial capital is x_j and an optimal strategy is followed with respect to consumption and investment

Y_j - present value at decision point j of the non-capital income stream capitalized at the borrowing rate of interest, i.e., $Y_j = \frac{y_j}{r_B} + \frac{y_{j+1}}{r_B^2} + \frac{y_{j+2}}{r_B^3} + \ldots$

By the boundedness of y_j, Y_j always exists

As stated in 1.3.1, consumption and investment decisions are assumed to be made at the beginning of each period. By the definition of β_{ij}, it is clear that i = 1 denotes the financial opportunities and that all other values of i denote the productive opportunities.

The amount allocated to consumption is assumed to be spent immediately or, if spent gradually over the period, to be set aside in a non-earning account. While no absolute limit will be placed on borrowing, it is assumed that no debt is forgivable and that the individual's borrowings must at all times be fully secured. This implies that the individual's debt cannot exceed the present value, on the basis of the borrowing rate of interest, of his non-capital income stream at the end of any period, and that there is an upper limit, given by $x_j + Y_j$, on consumption in any period j.

Throughout, the β_{ij} will be assumed to be independently distributed with respect to j. Except where otherwise indicated, we shall also assume that $r_B = r_L = r$.

In the initial models, we shall further assume that $y_j = y$, $M_j = M$, $S_j = S$ for all j, and that the β_{ij} are identically distributed with respect to j. The latter assumptions enable us to drop the subscript j since the decision problem is now the same at each decision point. Consequently, the employment of the last set of assumptions will be indicated by the absence of subscript j.

2.2 DERIVATION OF THE BASIC MODEL

As stated in Chapter I, the model to be constructed is probably closer, in contents, to Fisher's model of the individual, as presented in The Theory of Interest, than to any other. In form, the model may be viewed as a generalization of Phelps' personal savings model.[1]

We shall now identify the relation which determines the amount of capital (debt) on hand at each decision point in terms of the amount on hand at the previous decision point. This leads to the difference equation

$$(2\text{-}1) \qquad x_{j+1} = \sum_{i=2}^{M_j} \beta_{ij} z_{ij} + r z_{1j} + y_j \qquad\qquad j = 1, 2, \ldots$$

where

$$\sum_{i=1}^{M_j} z_{ij} = x_j - c_j \qquad\qquad j = 1, 2, \ldots$$

by direct application of the definitions given in 2.1. The first term of (2-1) represents the proceeds from productive investments, the

[1] Phelps, *op. cit.*

second term the payment of the debt or the proceeds from savings, and the third term the non-capital income received.

In order to eliminate the need for keeping track of the accounting equation which requires the z_{ij} to sum to $x_j - c_j$, we shall rewrite (2-1) slightly. To do this, deduct $\sum_{i=2}^{M_j} rz_{ij}$ from the first term on the right-hand side and add it to the second term. Utilizing the fact that $\sum_{i=1}^{M_j} z_{ij} = x_j - c_j$ for all j, we obtain

$$(2\text{-}2) \quad x_{j+1} = \sum_{i=2}^{M_j} (\beta_{ij} - r) z_{ij} + r(x_j - c_j) + y_j \qquad j = 1, 2, \ldots$$

This is the (difference) equation, then, which governs the process we are about to study.

The definition of $f_j(x)$ may formally be written

$$(2\text{-}3) \quad f_j(x_j) \equiv \max E[U(c_j, c_{j+1}, c_{j+2}, \ldots) | x_j]$$

From (1-5) we obtain, by the principle of optimality,[1] for all j

$$(2\text{-}4) \quad f_j(x_j) = \max E\left[\left\{u(c_j) + \alpha \max E[U(c_{j+1}, c_{j+2}, \ldots) | x_{j+1}]\right\} | x_j\right]$$

By (2-3), this reduces to

$$(2\text{-}5) \quad f_j(x_j) = \max\{u(c_j) + \alpha E[f_{j+1}(x_{j+1})]\} \qquad \text{all } j$$

[1]The principle of optimality states that an optimal strategy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal strategy with regard to the state resulting from the first decision. (See Richard Bellman, Dymanic Programming, Princeton, Princeton University Press, 1957, p. 83.

since, at decision point j+1, we are faced with the same kind of problem as when we are at j except that we now have a new capital position, x_{j+1}. Using (2-2), (2-5) becomes

$$(2\text{-}6) \quad f_j(x_j) = \max_{\substack{0 \le c_j \le x_j + Y_j \\ z_{ij} \ge 0 \\ \forall i \notin S_j}} \left\{ u(c_j) + \alpha E\left[f_{j+1}\left\{ \sum_{i=2}^{M_j} (\beta_{ij} - r) z_{ij} + r(x_j - c_j) + y_j \right\} \right] \right\} \quad j = 1, 2, \ldots$$

In the case when the β_{ij} are identically distributed with respect to j and the non-capital income stream is constant, (2-6) reduces to

$$(2\text{-}6a) \quad f(x) = \max_{\substack{0 \le c \le x+Y \\ z_i \ge 0 \\ \forall i \notin S}} \left\{ u(c) + \alpha E\left[f\left(\sum_{i=2}^{M} (\beta_i - r) z_i + r(x-c) + y \right) \right] \right\}$$

at each decision point.

For comparison, the model studied by Phelps is given by the functional equation

$$(2\text{-}6b) \quad f(x) = \max_{0 \le c \le x} \left\{ u(c) + \alpha E\left[f(\beta(x-c) + y) \right] \right\}$$

In this model, all capital not currently consumed obeys the transformation β, which is identically distributed in each period. Since the amount invested, x-c, is determined once c is known, (2-6b) has only one decision variable (c).

Since x represents capital, $f_j(x)$ is clearly the utility of money at the jth decision point. Instead of being assumed, as is generally the case, the utility function of money has in this model been induced from inputs which are more primitive than the preferences for money itself. As (2-6) shows, $f_j(x)$ depends on the individual's preferences with respect to consumption, the available investment opportunities and their riskiness, the interest rate, and his non-capital income stream. Are not these the very factors that an individual, given the task of constructing his utility of money, would consider? Since money is only a means to an end, it should therefore come as no surprise that its utility is dependent on the utility of the end and the opportunities for achieving it.

We shall now attempt to obtain the solution to (2-6) for certain classes of the function u(c). Since U is a cardinal utility function, it should be remembered that $\lambda_1 + \lambda_2 u(c)$, where λ_1 and $\lambda_2 > 0$ are constants, is also a utility function whenever u(c) is. To keep our expressions as simple as possible, we shall continue to use the simple representation u(c) in our analysis.

We shall now state and prove a preliminary result which will be needed later.

Lemma 1: Let u(c), β_i, i=2, ..., M, and r be defined as in 2.1 (when the subscript j is added). Then the function

$$(2\text{-}7)\quad h(v_2,v_3,\ldots,v_M) = E[u(\sum_{i=2}^{M} (\beta_i - r)v_i + r)]$$

subject to the feasibility constraint

$$(2\text{-}8)\quad \Pr\{\sum_{i=2}^{M} (\beta_i - r)v_i + r \geq 0\} = 1$$

and the constraint

(2-9) $v_i \geq 0$ for all $i \notin S$

has a maximum and the maximizing v_i ($\equiv v_i^*$) are finite and unique.

Proof: Since u(c) is undefined for c < 0, the purpose of (2-8) is to insure that the argument of u will be non-negative with probability 1. Let D_F be the (M-1)-dimensional space defined by the set of points $\tilde{v} \equiv (v_2, v_3, \ldots, v_M)$ which satisfy (2-8). Similarly, let D_S be the set of points $\tilde{v}$ which satisfy (2-9) and define $D \equiv D_F \cap D_S$. We shall first prove that h is strictly concave on the set D and that D itself is non-empty, closed, bounded, and convex.[1]

Differentiating (2-7) twice we obtain

(2-10) $$\frac{\partial h}{\partial v_i} = E[u'(\sum_{i=2}^{M} (\beta_i - r)v_i + r)(\beta_i - r)] \qquad i = 2, \ldots, M$$

(2-11) $$\frac{\partial^2 h}{\partial v_i^2} = E[u''(\sum_{i=2}^{M} (\beta_i - r)v_i + r)(\beta_i - r)^2] \qquad i = 2, \ldots, M$$

Then, since $u''(c) < 0$ for all $c \geq 0$ by the strict concavity of u, $(\beta_i - r)^2 \geq 0$, and $\Pr\{\beta_i - r \neq 0\} > 0$, by the "no-easy-money condition" (see 2.1), we find that

(2-12) $$\frac{\partial^2 h}{\partial v_i^2} < 0 \text{ for all } i$$

whenever $\tilde{v} \in D_F$. Thus, h is strictly concave on the set D.

[1]The author gratefully acknowledges a debt to Professor Brown for several valuable suggestions concerning the proof of the closure and the boundedness of D.

The non-emptiness of D follows trivially from the observation that $\bar{v}^0 \equiv (0,0,\ldots,0)$ is a member of D. By the boundedness of the β_i's and of r (properties 1 and 2 in 2.1), there exists a neighborhood of v^0 in relation to D. That is, there is a neighborhood of points $\bar{v}'$ such that

$$\Pr\{\sum_{i=2}^{M} (\beta_i - r)v_i' + r \geq 0\} = 1$$

where $v_i' \geq 0$ for all $i \notin S$

Now consider the point $\bar{v}^\lambda \equiv \bar{v}^0 + \lambda\bar{v}' = \lambda v'$ where $\lambda \geq 0$ and v' is one of the points in this neighborhood. Let b(v) be the greatest lower bound on b such that

$$\Pr\{\sum_{i=2}^{M} (\beta_i - r)v_i < b\} > 0$$

By the "no-easy-money condition" of 2.1, $b(\bar{v}') \geq -r$ for $v' \in D$, $b(\bar{v}^0) = 0$, and $b(\bar{v}) < 0$ for all $\bar{v} \neq \bar{v}^0$. Applying the "no-easy-money condition" with respect to the point $\bar{v}^\lambda$, we obtain, since we may write

$$\Pr\{\sum_{i=2}^{M} (\beta_i - r)\lambda\bar{v}_i < \lambda b\} > 0$$

that $\lambda b(\bar{v}') = b(\lambda\bar{v}')$. But when $\lambda b(\bar{v}') < -r$, or $\lambda > -r/b(\bar{v}')$, the point $\bar{v}^\lambda$ cannot lie in D since $\lambda > -r/b(\bar{v}')$ implies that

$$\Pr\{\sum_{i=2}^{M} (\beta_i - r)\lambda v_i' + r \geq 0\} < 1$$

Thus, $\lambda_0 = -r/b(\bar{v}')$ is the greatest lower bound on λ such that $\bar{v}^\lambda \notin D$. Since $\lambda_0 b(\bar{v}') = -r$, $\bar{v}^{\lambda_0} \in D$ and is in fact the point farthest from $\bar{v}^0$ lying on the line through $\bar{v}^0$ and $\bar{v}'$ and belonging to D.

We shall only sketch the remainder of the proof establishing the closure and boundedness of D. Let $\bar{v} \neq \bar{v}^0$ be the limit point of a sequence of points $\{\bar{v}^{(n)}\} \in D$. Since each point in the sequence belongs to D, $b(\bar{v}^{(n)}) \geq -r$ for all n. It can now be shown, by utilizing the fact that $\sum_{i=2}^{M} (\beta_i - r)v_i$ is continuous at any $\bar{v} \neq \bar{v}^0$, uniformly with respect to the β_i's on any bounded set, that $\overline{\lim_{n\to\infty}}\, b(\bar{v}^{(n)}) \leq b(\bar{v})$, which implies that $\bar{v} \in D$. Consequently, D must be closed.

The boundedness of D is established as follows. Let S_R be the set of points $\bar{v}$ such that $|\bar{v}| = R > 0$. S_R is then clearly both closed and bounded. If $D' \equiv D \cap S_R$ is empty, the boundedness of D follows immediately. Let us therefore assume that D' is non-empty; in this case D' is also bounded and closed since D is closed and S_R is bounded and closed. If $\bar{v}$ is a limit point of the sequence $\{\bar{v}^{(n)}\}$ such that $\bar{v}^{(n)} \in D'$, we must have that $\bar{v} \in D'$ since D' is closed. But $b(\bar{v}) < 0$ by the "no-easy-money condition" (see 2.1) since $\bar{v} \neq v^0$ by assumption. Therefore, since we already have that $\overline{\lim_{n\to\infty}}\, b(\bar{v}^{(n)}) \leq b(\bar{v})$, 0 cannot be a limit point to the sequence $\{b(\bar{v}^{(n)})\}$, $\bar{v}^{(n)} \in D'$. Consequently, $b(\bar{v})$ for $\bar{v} \in D'$ is bounded away from zero, which implies that D must be bounded.

To prove convexity, let $\bar{v}''$ and $\bar{v}'''$ be two points in D. Then, for any $0 \leq \lambda \leq 1$,

$$\Pr\{\sum_{i=2}^{M} (\beta_i - r)\lambda v_i'' + \lambda r \geq 0\} = 1$$

and

$$\Pr\{\sum_{i=2}^{M}(\beta_i - r)(1-\lambda)v_i''' + (1-\lambda)r \geq 0\} = 1$$

which implies

$$\Pr\{\sum_{i=2}^{M}(\beta_i - r)(\lambda v_i'' + (1-\lambda)v_i''') + r \geq 0\} = 1$$

so that $\lambda\bar{v}'' + (1-\lambda)\bar{v}''' \in D$. Thus, D is convex.

Since our problem has now been shown to be one of maximizing a strictly concave function over a non-empty, closed, bounded, convex set, it follows from the Kuhn-Tucker Theorem that the function h has a maximum and that the v_i^* are finite and unique.[1]

A number of corollaries follow from this lemma which we shall also find useful later.

Corollary 1. Let $u(c)$, β_i, $i = 2, \ldots, M$, and r be defined as in Lemma 1. Moreover, let $u(c)$ be such that it has no lower bound. Then the v_i^* which maximize (2-7) are such that $\Pr\{\sum_{i=2}^{M}(\beta_i - r)v_i^* + r > 0\} = 1$.

The proof is immediate from the observation that $h \to -\infty$ as the greatest lower bound on b such that $\Pr\{\sum_{i=2}^{M}(\beta_i - r)v_i + r < b\} > 0$ approaches 0 from above.

Corollary 2. Let $u(c)$, β_i, $i = 2, \ldots, M$ and r be defined as in Lemma 1. Then the maximum of the function (2-7) subject to the constraints (2-8) and (2-9) is greater than or equal to $u(r)$.

[1] H. W. Kuhn and A. W. Tucker, "Nonlinear Programming," Second Berkeley Symposium on Mathematical Statistics and Probability, Berkeley, University of California Press, 1951, pp. 481-486.

Proof: When $v_i = 0$ for all i, we obtain by (2-7) $h = u(r)$. Unless some other feasible v_i can make $h > u(r)$, $v_i^* = 0$ since the zero solution is always feasible.

Corollary 3. Let $u(c)$, β_i, $i = 2, \ldots, M$, and r be defined as in Lemma 1. Moreover, let $u(c)$ be such that $u(c) \leq b$. Then the v_i which satisfy (2-8) and (2-9) are such that

$$\max E[u(\sum_{i=2}^{M} (\beta_i - r)v_i + r)] < b$$

The proof is immediate from Lemma 1.

Corollary 4. Let $u(c)$, β_i, $i = 2, \ldots, M$, and r be defined as in Lemma 1. Moreover, let $u(c)$ be such that it has no lower bound, let S be such that all $i \in S$, and let the β_i be independently distributed. Then the v_i^* which maximize (2-7) subject to (2-8) (and (2-9)) are such that

$$\text{(2-13)} \quad v_i^* \gtreqless 0 \text{ if and only if } E[\beta_i] \gtreqless r \qquad i = 2, \ldots, M.$$

Proof: Since the β_i are statistically independent, the partial derivative (2-10), evaluated at $v_i = 0$, may be written

$$\frac{\partial h}{\partial v_i}(v_i=0) = E[u'(\sum_{\substack{k=2 \\ k \neq i}}^{M} (\beta_k - r)v_k + r)]E[\beta_i - r]$$

But

$$\frac{\partial h}{\partial v_i}(v_i=0) \gtreqless 0 \text{ as } E[\beta_i] \gtreqless r$$

since the first factor is positive from the monotonicity of u. By

(2-12), this proves the first part of (2-13) since the assumptions with respect to u(c) and S insure an interior maximum with respect to (2-8) (see Corollary 1), and (2-9) is non-existent. Again by (2-12), if

$v_i \gtreqqless 0$ when $\frac{\partial h}{\partial v_i} \gtreqqless 0$, then $\frac{\partial h}{\partial v_i}(v_i=0) \gtreqqless 0$, giving $E[\beta_i] \gtreqqless r$.

Lemma 2. Let u(c), β_i, i = 2, ..., M, and r be defined as in 2.1 except that u(c) has an upper bound and is defined for c < 0. Moreover, let $r_1 \geq 0$ be a constant. Then the function

$$(2\text{-}14) \quad \bar{h}(v_2,\ldots,v_M) = E[u(\sum_{i=2}^{M}(\beta_i - r)v_i + r_1)]$$

where $v_i \geq 0$ for all $i \notin S$ has a maximum and the maximizing $v_i (\equiv v_i^*)$ are finite and unique.

The proof is similar to that of Lemma 1. Corollaries 2-4 also hold with (2-7) replaced by (2-14).

2.3 THE SOLUTION WHEN u(xy) = u(x)|u(y)|

We shall first consider the class of utility functions u(c) such that u(xy) = u(x)|u(y)|. This class consists of the functions[1]

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[1]The set of solutions to the functional equation u(xy) = u(x)u(y) is given by (See J. Aczél and S. Golass, Funktionalgleichungen der Theorie der Geometrischen Objekte, Warsaw, Panstwowe Wydawnictwo Naukowe, 1960, pp. 102-103.)

(A) $u(x) \equiv 0$

(B) $u(x) = |x|^\gamma$

(C) $u(x) = \text{sg } x$

(D) $u(x) = |x|^\gamma \text{sg } x \ (\gamma \neq 0)$

where γ is a constant. However, under our restrictions (u(x) monotone increasing and strictly concave, x ≥ 0), but including the possibility that u(xy) = u(x)|u(y)|, the set (A)-(D) reduces to (2-15) and (2-16).

(2-15) $u(c) = c^{\gamma} \qquad 0 < \gamma < 1$

and

(2-16) $u(c) = -c^{-\gamma} \qquad \gamma > 0$

As in the following, we shall employ the method of successive approximations in seeking the solution to (2-6a). Let the Nth approximation of f(x) and the optimal strategies c(x) and $z_i(x)$, $i = 1, \ldots, M$; be denoted $f_N(x)$, $c_N(x)$, and $z_i^{(N)}(x)$, respectively, and define

$$t_N = x + \frac{y}{r} + \frac{y}{r^2} + \ldots \frac{y}{r^{N-1}}$$

We then obtain

$$\text{(2-17)} \quad f_N(x) = \max_{\substack{0 \le c_N \le t_N \\ z_i^{(N)} \ge 0 \\ \forall i \notin S}} \left\{ u(c_N) + \alpha E[f_{N-1}\{ \sum_{i=2}^{M} (\beta_i - r) z_i^{(N)} + r(x-c_N) + y\}] \right\}$$

where $f_1(x) \equiv u(x)$.

Solving successively, we get

$$f_2(x) = \max_{\substack{0 \le c_2 \le t_2 \\ z_i^{(2)} \ge 0 \\ \forall i \notin S}} \left\{ u(c_2) + \alpha E[f_1(\sum_{i=2}^{M} (\beta_i - r) z_i^{(2)} + r(x-c_2) + y)] \right\}$$

$$= \max \left\{ u(c_2) + \alpha E[u(\sum_{i=2}^{M} (\beta_i - r) z_i^{(2)} + r(x-c_2) + y)] \right\}$$

$$= \max\left\{u(c_2) + \alpha|u(t_2-c_2)|E[u(\sum_{i=2}^{M}(\beta_i-r)\frac{z_i^{(2)}}{t_2-c_2} + r)]\right\}$$

By Lemma 1, the function

(2-18) $$h(v_2, \ldots, v_M) = E[u(\sum_{i=2}^{M}(\beta_i-r)v_i + r)]$$

subject to (2-8) and (2-9) has a unique maximum for finite v_i, $i = 2, \ldots, M$. Denoting the maximizing values by v_i^* and the maximum of (2-18) by k, we note that the set of values

(2-19)
$$z_i^{(2)}(x) = (t_2 - c_2)v_i^* \qquad i = 2, \ldots, M$$

$$z_1^{(2)}(x) = x - c_2 - \sum_{i=2}^{M} z_i^{(2)}(x)$$

maximizes h for <u>all</u> values of x and c_2. Since

(2-20) $$\frac{\partial f_2}{\partial h} = \alpha|u(t_2-c_2)| \geq 0$$

we clearly wish to make h as large as possible. Thus, the investment strategy given by (2-19) is optimal.

Differentiating with respect to c_2 (after maximizing with respect to $z_i^{(2)}$ for all i), we obtain, since $u(t_2 - c_2)\alpha|k| = |u(t_2 - c_2)|\alpha k$ by (2-15) and (2-16),

(2-21) $$\frac{\partial f_2}{\partial c_2} = u'(c_2) - \alpha|k|u'(t_2 - c_2)$$

By the monotonicity and strict concavity of $u(c)$, $u'(c)$ is monotone decreasing; it therefore has an inverse. Denote the inverse of $\frac{1}{\gamma} u'$ by g, where γ is the positive constant in (2-15) or (2-16), i.e.,

$$(2\text{-}22) \quad g\left(\frac{1}{\gamma} u'(c)\right) = c$$

It then follows that g is positive and decomposable in the same way as (2-15), i.e., $g(xy) = g(x)g(y)$. Upon setting the partial derivative (2-21) equal to zero, we obtain

$$(2\text{-}23) \quad c_2 - g(\alpha|k|)(t_2 - c_2) = 0$$

so that

$$(2\text{-}24) \quad c_2(x) = \frac{g(\alpha|k|)}{1 + g(\alpha|k|)} t_2$$

where $c_2(x)$ clearly lies in the interval $[0, t_2]$. The maximum condition follows from the concavity of f_2 in c_2 (to be shown later). $f_2(x)$ now becomes

$$(2\text{-}25) \quad f_2(x) = \left| u\left[\frac{g(\alpha|k|)}{1 + g(\alpha|k|)}\right] \right| u(t_2) + \alpha|k| u\left[t_2 - \frac{g(\alpha|k|)}{1 + g(\alpha|k|)} t_2\right]$$

$$= u(t_2) \left| u\left[\frac{g(\alpha|k|)}{1 + g(\alpha|k|)}\right] \right| \left[1 + \left| u\left\{\frac{1}{g(\alpha|k|)}\right\} \right| \alpha|k|\right]$$

$$= u(t_2) \left| u\left[\frac{g(\alpha|k|)}{1 + g(\alpha|k|)}\right] \right| \left[1 + \frac{|u(1)|}{g(\alpha|k|)}\right]$$

$$= K_2 u(t_2) \qquad (K_2 \text{ constant})$$

since $|u(g(\alpha|k|))| = \alpha|k|g(\alpha|k|)$ from $1/\lambda u'(g(\alpha|k|)) = \alpha|k|$

Continuing in this fashion, we obtain, for N = 2, 3, ...

$$(2\text{-}26)\quad f_N(x) = u(t_N)\left|u\left[\frac{g[(\alpha|k|)^{N-1}]}{1 + g(\alpha|k|) + g[(\alpha|k|)^2] + \ldots + g[(\alpha|k|)^{N-1}]}\right]\right|$$

$$\left\{1 + \frac{|u(1)|}{g(\alpha|k|)} + \ldots + \left[\frac{|u(1)|}{g(\alpha|k|)}\right]^{N-1}\right\}$$

$$(2\text{-}27)\quad c_N(x) = \frac{g[(\alpha|k|)^{N-1}]}{1 + g(\alpha|k|) + \ldots + g[(\alpha|k|)^{N-1}]}\, t_N$$

$$z_i^{(N)}(x) = (t_N - c_N)v_i^* \qquad i = 2, \ldots, M$$

$$(2\text{-}28)\quad z_1^{(N)}(x) = x - c_N - \sum_{i=2}^{M} z_i^{(N)}(x)$$

Denoting $\lim_{N\to\infty} f_N(x)$ by f(x), we find that f(x) exists whenever

$$(2\text{-}29)\quad g(\alpha|k|) > \max\{1, |u(1)|\}$$

It is then given by

$$(2\text{-}30)\quad f(x) = u(t)\left|u\left[\frac{g(\alpha|k|)-1}{g(\alpha|k|)}\right]\right|\left[\frac{g(\alpha|k|)}{g(\alpha|k|)-|u(1)|}\right]$$

$$= Ku(t) \qquad\qquad (K\ \text{constant})$$

where $t = \lim_{N\to\infty} t_N = x + \frac{y}{r-1} = x + \gamma$.

Furthermore $c_N(x)$ and $z_i^{(N)}(x)$, $i = 1, \ldots, M$, converge to

(2-31) $c(x) = [1 - g(\alpha|k|)^{-1}]t$

$$z_i(x) = (t - c)v_i^* \qquad i = 2, \ldots, M$$

(2-32)

$$z_1(x) = x - c - \sum_{i=2}^{M} z_i(x)$$

We must now show that the solution is unique. Since the sum of two strictly concave functions is strictly concave, $\max_{c_2} \{u(c_2) + \alpha|k|u(t_2-c_2)\}$ is strictly concave for $0 \le c_2 \le t_2$, and therefore for $0 \le c_2 \le x + y/r$, by the strict concavity of $u(c)$.[1] Therefore, by (2-21), $f_2(x)$ is strictly concave for $x \ge -y/r$. As a result, $c_2(x)$ is unique. By (2-19), $z_i^{(2)}(x)$, $i = 1, \ldots, M$, is then also unique. By induction, we obtain that each function in the sequence $\{f_N(x)\}$ is strictly concave and that the sequences $\{c_N(x)\}$, $\{z_i^{(N)}(x)\}$, $i = 1, \ldots, M$, are unique. Thus, the limit function $f(x)$ is strictly concave. From this it follows that the optimal strategies are unique.

The preceding now establishes

Theorem 2. Let α, $u(c)$, β_i, $i = 2, \ldots, M$, r, y, and $f(x)$ be defined as in 2.1. Moreover, let $u(c)$ be such that $u(xy) = u(x)|u(y)|$ and

[1] The proof of the theorem which states that if $G(x,y)$ is a strictly concave function of x and y for $x,y \ge 0$, then the function $H(x)$ defined by $H(x) = \max_{0 \le y \le x} G(x,y)$ is strictly concave in x for $x \ge 0$ may be found in Richard Bellman, op. cit., p. 21.

let the function g be the inverse of $(1/\lambda)u'(c)$, where λ is the constant in (2-15) or (2-16). Then a solution to (2-6a) exists for $x \geq -Y$ whenever $g(\alpha|k|) > \max\{1, |u(1)|\}$ and is given by (2-30)-(2-32), where k is the maximum of (2-18) (subject to (2-8) and (2-9)) and the v_i^* are the values of v_i which give the maximum. Furthermore, the optimal strategies (2-31) and (2-32) are unique.

When y = 0, the solution to (2-6a) reduces to

$$f(x) = Ku(x)$$

$$c(x) = [\,1 - g(\alpha|k|)^{-1}]x$$

$$z_i(x) = (x-c)v_i^* \qquad i = 2, \ldots, M$$

$$z_1(x) = x - c - \sum_{i=2}^{M} z_i(x)$$

But then, letting t = x + Y as before,

$$f(t) = Ku(t)$$

$$c(t) = [\,1 - g(\alpha|k|)^{-1}]t$$

$$z_i(t) = (t-c)v_i^* \qquad i = 2, \ldots, M$$

$$z_1(t) = t - c - \sum_{i=2}^{M} z_i(t)$$

As a result, except for $z_1(x + Y)$, the solution to the original problem is not altered when the individual, instead of receiving the non-capital income stream in installments, is given its present value Y in advance. Thus, instead of letting x be the state variable when there is a non-capital income, one could let $x + Y$ be the state variable (pretending there is no income), as long as Y is deducted from $z_1(x + Y)$.

2.3.1 Model I

As was indicated earlier, one of the two possible functions which satisfy the decomposability requirement of Theorem 2 is

$$(2\text{-}15) \quad u(c) = c^{\lambda} \qquad 0 < \lambda < 1$$

We shall refer to the decision problem (2-6) when $u(c)$ is of this form as Model I. The chief characteristic of (2-15) with which we shall be concerned is that $u(c)$ has a lower bound but no upper bound.

When (2-15) holds, $u'(c) = \lambda c^{\lambda -1}$ so that $g(x) = x^{\frac{-1}{1-\lambda}}$. Thus the convergence condition (2-29) becomes $(\alpha|k|)^{\frac{-1}{1-\lambda}} > 1$, or $\alpha|k| < 1$. By Corollary 2, $k \geq r^{\lambda}$, so that $\alpha k > 0$ and $f(x)$ is finite only for $\alpha < 1/r^{\lambda} < 1$. Thus, when $\alpha k < 1$, the solution to Model I for $x \geq -Y$ is, by Theorem 2,

$$(2\text{-}33) \quad f(x) = \left[\frac{(\alpha k)^{\frac{-1}{1-\lambda}}}{(\alpha k)^{\frac{-1}{1-\lambda}} - 1}\right]^{1-\lambda} (x + Y)^{\lambda}$$

$$(2\text{-}34) \quad c(x) = \left[1 - (\alpha k)^{\frac{1}{1-\lambda}}\right](x + Y)$$

$$z_i(x) = (x + Y - c)v_i^* \qquad i = 2, \ldots, M$$

(2-35)

$$z_1(x) = x - c - \sum_{i=2}^{M} z_i(x)$$

where k and the v_i^* are given by

$$\text{(2-36)} \quad k \equiv E\left[\{\sum_{i=2}^{M}(\beta_i - r)v_i^* + r\}^{\lambda}\right]$$

$$= \max E\left[\{\sum_{i=2}^{M}(\beta_i - r)v_i + r\}^{\lambda}\right]$$

subject to (2-8) and (2-9).

2.3.2 <u>Model II</u>

Let us now examine the second class of functions u for which Theorem 2 holds, namely

$$\text{(2-16)} \quad u(c) = -c^{-\lambda} \qquad \lambda > 0$$

Here, u(c) has an upper bound but no lower bound. Since $u'(c) = \lambda c^{-\lambda-1}$, $g(x) = x^{\frac{-1}{\lambda+1}}$ so that the convergence condition (2-29) becomes $(\alpha|k|)^{\frac{-1}{\lambda+1}} > 1$, or $\alpha|k| < 1$. By Corollary 2, $k \geq -r^{-\lambda}$, and by Corollary 3, $k < 0$; thus $|k| \leq r^{-\lambda} < 1$ so that the convergence condition always holds.

By Theorem 2, the solution to Model II is then, for $x \geq -Y$

$$(2\text{-}37)\quad f(x) = \left[\frac{(\alpha(-k))^{\frac{-1}{\lambda+1}}}{(\alpha(-k))^{\frac{-1}{\lambda+1}}-1}\right]^{\lambda+1} [-(x+Y)^{-\lambda}]$$

$$(2\text{-}38)\quad c(x) = \left[1 - (\alpha(-k))^{\frac{1}{\lambda+1}}\right](x+Y)$$

$$z_i(x) = (x + Y - c)v_i^* \qquad i = 2, \ldots, M$$

(2-39)

$$z_1(x) = x - c - \sum_{i=2}^{M} z_i(x)$$

where k and v_i^* are given by

$$(2\text{-}40)\quad k \equiv E\left[-(\sum_{i=2}^{M}(\beta_i - r)v_i^* + r)^{-\lambda}\right]$$

$$= \max E\left[-(\sum_{i=2}^{M}(\beta_i - r)v_i + r)^{-\lambda}\right]$$

subject to (2-8) and (2-9).

2.4 THE SOLUTION WHEN u(xy) = u(x) + u(y)

We shall now consider the class of utility functions which is decomposable in such a way that u(xy) = u(x) + u(y). Since only one function satisfies this property, namely u(c) = log c, we shall therefore replace u(c) with log c in (2-6) whenever this class is considered.[1] It should be noted that this class of utility functions

[1] See J. Aczél, Vorlesungen über Funktionalgleichungen und Ihre Anwendungen, Stuttgart, Birkhauser Verlag, 1961, p. 48.

has neither an upper nor a lower bound. The solution is now given by

Theorem 3. Let α, β_i, $i = 2, \ldots, M$, r, y, and f(x) be defined as in 2.1. Moreover, let u(c) = log c. Then a solution to (2-6a) exists for $x \geq -Y$ and is given by

$$(2\text{-}41)\quad f(x) = \frac{1}{1-\alpha}\log(x+Y) + \frac{1}{1-\alpha}\log(1-\alpha) + \frac{\alpha \log \alpha}{(1-\alpha)^2} + \frac{\alpha k}{(1-\alpha)^2}$$

$$(2\text{-}42)\quad c(x) = (1-\alpha)(x+Y)$$

$$(2\text{-}43)\quad \begin{aligned} z_i(x) &= (x+Y-c)v_i^* \qquad\qquad i = 2, \ldots, M \\ z_1(x) &= x - c - \sum_{i=2}^{M} z_i(x) \end{aligned}$$

where k and v_i^* are given by

$$(2\text{-}44)\quad \begin{aligned} k &= E\left[\log\left(\sum_{i=2}^{M}(\beta_i - r)v_i^* + r\right)\right] \\ &= \max E\left[\log\left(\sum_{i=2}^{M}(\beta_i - r)v_i + r\right)\right] \end{aligned}$$

subject to (2-8) and (2-9). Furthermore, the optimal strategies (2-42) and (2-43) are unique.

Proof: Let us verify that the solution satisfies (2-6a) by denoting the right-hand side T(x) upon inserting (2-41) for f(x). Then

$$T(x) = \max_{\substack{0 \leq c \leq x+Y \\ z_i \geq 0 \\ \forall i \notin S}} \left\{\log c + \frac{\alpha}{1-\alpha} E\left[\log\left(\sum_{i=2}^{M}(\beta_i - r)z_i\right.\right.\right.$$

$$+ r(x-c) + y + \frac{y}{r-1})\Big]\Big\}$$

$$+ \frac{\alpha}{1-\alpha} \log(1-\alpha) + \frac{\alpha^2 \log \alpha}{(1-\alpha)^2} + \frac{\alpha^2 k}{(1-\alpha)^2}$$

$$= \max_{\substack{0 \le c \le x+Y \\ z_i \ge 0 \\ \forall i \notin S}} \Big\{\log c + \frac{\alpha}{1-\alpha} \log(x+Y-c)$$

$$+ \frac{\alpha}{1-\alpha} E\Big[\log\{\sum_{i=2}^{M} (\beta_i - r)\frac{z_i}{x+Y-c} + r\}\Big]\Big\}$$

$$+ \frac{\alpha}{1-\alpha} \log(1-\alpha) + \frac{\alpha^2 \log \alpha}{(1-\alpha)^2} + \frac{\alpha^2 k}{(1-\alpha)^2}$$

By (2-7), the third term can be written

$$\frac{\alpha}{1-\alpha} h\left[\frac{z_2}{x+Y-c}, \ldots, \frac{z_M}{x+Y-c}\right]$$

where $\frac{\partial T}{\partial h} > 0$. Since the maximum of h is k by (2-44) and strategy (2-43) assures this value regardless of the values of x and of c, (2-43) is clearly the maximizing strategy. The third term then becomes $\frac{\alpha k}{1-\alpha}$. Solving

$$\frac{\partial T}{\partial c} = \frac{1}{c} - \frac{\alpha}{(1-\alpha)(x+Y-c)} = 0$$

for c we obtain (2-42). Thus,

$$T(x) = \log(1-\alpha) + \log(x+Y) + \frac{\alpha}{1-\alpha}\log\alpha + \frac{\alpha}{1-\alpha}\log(x+Y)$$

$$+ \frac{\alpha k}{1-\alpha} + \frac{\alpha}{1-\alpha}\log(1-\alpha) + \frac{\alpha^2\log\alpha}{(1-\alpha)^2} + \frac{\alpha^2 k}{(1-\alpha)^2}$$

$$= \frac{1}{1-\alpha}\log(x+Y) + \frac{1}{1-\alpha}\log(1-\alpha) + \frac{\alpha\log\alpha}{(1-\alpha)^2} + \frac{\alpha k}{(1-\alpha)^2}$$

$$= f(x)$$

The uniqueness of the solution follows immediately from the strict concavity of $f(x)$.

2.5 THE SOLUTION WHEN $u(x + y) = u(x)|u(y)|$

We shall now examine the case when $u(x + y) = u(x)|u(y)|$. The only utility functions, by the criteria of 2.1, which satisfy this functional equation are those given by[1]

(2-45) $u(c) = -e^{-\gamma c}$ $\qquad \gamma > 0$

Since $-1 \le u(c) \le 0$, this class provides an example of a utility function with both an upper and a lower bound. The solution to (2-6a) is now given by

Theorem 4. Let α, β_i, $i = 2, \ldots, M$, r, y, and $f(x)$ be defined as in 2.1. Moreover, let $u(c) = -e^{-\gamma c}$ for $c \geq 0$ where $\gamma > 0$. Then the solution to (2-6a) exists for $x \ge -Y+[r/(\gamma(r-1)^2)]\log(-\alpha kr)$ and is given by

(2-46) $$f(x) = -\frac{r}{r-1}(-\alpha kr)^{\frac{r}{r-1}} e^{-\frac{\gamma(r-1)}{r}(x+Y)}$$

[1]The solution to the functional equation $u(x+y) = u(x)|u(y)|$ is $u(x) = e^{\gamma x}$, $u(x) = -e^{\gamma x}$, $u(x) \equiv 0$ (see J. Aczél, Vorlesungen . . ., pp. 47-48). Of these, only the subset given by (2-45) is strictly concave and monotone increasing.

$$(2\text{-}47)\quad c(x) = \frac{r-1}{r}(x + Y) - \frac{1}{\gamma(r-1)}\log(-\alpha kr)$$

$$z_i(x) = \frac{r}{\gamma(r-1)} v_i^* \qquad\qquad i = 2, \ldots, M$$

(2-48)

$$z_1(x) = x - c - \sum_{i=2}^{M} z_i(x)$$

where k and v_i^* are given by

$$(2\text{-}49)\quad k \equiv E[-e^{-\sum_{i=2}^{M}(\beta_i - r)v_i^*}]$$

$$= \max_{\substack{v_i \geq 0 \\ \forall i \notin S}} E[-e^{-\sum_{i=2}^{M}(\beta_i - r)v_i}]$$

<u>provided</u> that

$$(2\text{-}50)\quad \log(-\alpha kr) + b(\bar{v}^*) \geq 0$$

where $b(\bar{v}^*)$ is the greatest lower bound on b such that

$$\Pr\{\sum_{i=2}^{M}(\beta_i - r)v_i^* < b\} > 0$$

and $\bar{v}^* \equiv (v_2^*, \ldots, v_M^*)$. Moreover, the optimal strategies (2-47) and (2-48) are unique.

<u>Proof</u>: To obtain this solution, it is necessary to proceed in two steps. First, consider (2-6a) with the non-negativity restriction on c removed. The new equation is still well defined mathematically

since the function (2-45) is defined for all c. We have simply removed the economic content of u(c), and hence (2-6a). However, if, upon solving the new equation, we can find an interval of x such that, once entered, c(x) will remain non-negative with probability 1, the economic content may be reimputed to the solution for that interval. Thus, the second step is to find the conditions, if any, which guarantee the non-negativity of c(x).

Step 1: By a proof similar to that of Theorem 3, it can be shown that the solution to (2-6a), under the assumptions of Theorem 4 but with the restriction on c removed, exists for all x and is given by (2-46)-(2-49), the optimal strategies (2-47) and (2-48) being unique.

Step 2: Upon examination of (2-47), it is readily seen that $c(x) \geq 0$ whenever

$$(2\text{-}51) \quad x \geq -Y + \frac{r}{\gamma(r-1)^2} \log(-\alpha kr)$$

since c is strictly increasing in x. By (2-2), it follows that the greatest lower bound on the capital position at the beginning of the next period is

$$(2\text{-}52) \quad b(\bar{z}) + r(x-c) + y$$

Deducting the current capital x, the minimum increase in each period becomes

$$(2\text{-}53) \quad \frac{r}{\gamma(r-1)} [b(\bar{v}^*) + \log(-\alpha kr)]$$

once the optimal strategies (2-47) and (2-48) are inserted in (2-52). Whenever (2-53) is non-negative, it follows that the probability of a capital decrease is 0. Since (2-53) is independent of x, and in fact a constant, we obtain that capital will never decrease, regardless of the value of x, when (2-53) holds. Since $r/(\gamma(r-1)) > 0$, $c(x) \geq 0$ whenever (2-51) and (2-50) hold. When (2-50) holds, we also find that the right-hand side of (2-51) is greater than or equal to -Y, which gives the theorem.

Since the second term in (2-50) is always non-positive, it follows that it is necessary, but not sufficient, for (2-50) to hold that $-\alpha kr \geq 1$. (By Lemma 2 and Corollaries 2 and 3, $-1 \leq k < 0$; thus, it is also necessary, but not sufficient, that $\alpha \geq 1/r$.)

2.6 PROPERTIES OF THE OPTIMAL CONSUMPTION STRATEGIES

In each of the four models we note that the optimal consumption function c(x) is linear increasing in capital x and in non-capital income y. Whenever $y > 0$, positive consumption is called for even when the individual's net worth is negative, as long as it is greater than -Y in Models I-III and greater than $-Y + [r/(\gamma(r-1)^2)]\log(-\alpha kr)$ in Model IV. Only at these end points would the individual consume nothing.

The optimal consumption strategies have an interesting relation to the consumption hypotheses of Modigliani and Brumberg[1] and of

[1] F. Modigliani and R. Brumberg, "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data," Post-Keynesian Economics (ed. K. Kurihara), New Brunswick, Rutgers University Press, 1954.

Friedman,[1] which form an important part of the so-called new consumption theories.[2] One of the hypotheses, usually referred to as the normal income hypothesis or as the permanent income hypothesis, essentially states that an individual's consumption in any period depends only on his normal (permanent) income. Normal income is usually taken to include the current value of the individual's net assets plus the present value of his future non-capital income stream. The second hypothesis, named the proportionality hypothesis, states that "for any individual, the relationship between his consumption and his normal (permanent) income is one of proportionality."[3] Both of these hypotheses have been at least partially confirmed by extensive empirical tests carried out by their sponsors.

Surprisingly, the optimal consumption strategies of Models I-III satisfy the properties specified by the two consumption hypotheses precisely. Thus, individuals who maximize expected utility from consumption over time in a risky environment and whose preferences are of the class defined by Models I-III exhibit the same kind of behaviour with respect to consumption that characterizes a great many people in the real world. While this significant meeting-point between descriptive and normative consumption theory has a number of interesting aspects, the most significant observation from the point of view of this study is that the class of utility functions (1-5) such that $u(c) = c^{\gamma}$, $0 < \gamma < 1$, $u(c) = -c^{-\gamma}$, $\gamma > 0$, or $u(c) = \log c$ may

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[1] Milton Friedman, A Theory of the Consumption Function, Princeton, Princeton University Press, 1957.

[2] M. J. Farrell, "The New Theories of the Consumption Function," Economic Journal, December 1959.

[3] Ibid., p. 681.

represent a valid approximation of the preferences of a large group of individuals.

2.6.1 Effect of Impatience Rate

We shall now examine the effect of impatience on the optimal consumption strategies. Since the functions (2-34), (2-38), (2-42), and (2-47) are all decreasing in α, we find in each case that the greater the individual's impatience $1-\alpha$ is, the greater his present consumption would be. This, of course, is what we would expect.

2.6.2 Effect of Risk Aversion Index

Pratt proposes as a measure of the risk aversion possessed by a utility indicator $u(c)$ the function[1]

$$(2\text{-}54) \quad q(c) = -\frac{u''(c)}{u'(c)}$$

In Model IV, we find that $q(c) = \gamma$; since $c(x)$ is seen to be increasing in γ, the greater the risk aversion of $u(c)$, the greater the amount of present consumption. It should be noted that (2-45) is the <u>only</u> strictly concave function,[2] and hence the only utility function, for which the risk aversion index (2-54) is constant for all c.

Pratt also defines a second function

$$q^*(c) = cq(c)$$

which he calls the proportional risk aversion index.[1] By this measure

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[1] John Pratt, "Risk-Aversion in the Small and in the Large," Econometrica, January-April 1964, p. 122.

[2] Ibid., p. 130.

we obtain

(2-55) $q^*(c) = 1-\gamma$

(2-56) $q^*(c) = \gamma+1$

(2-57) $q^*(c) = 1$

for Models I, II, and III, respectively. By reference to (2-34) and (2-38), we find that $c(x)$ is increasing in $q^*(c)$. Again, the one-period utility functions of Models I-III are the only ones for which $q^*(c)$ is constant.[1] Thus, the more risk aversive the individual's consumption preferences are, the more he will favor the present at the expense of the future. Note that the preceding statements say nothing about his investment behavior but refer only to his optimal consumption pattern.

2.6.3 Effect of the "Favorableness" of the Investment Opportunities

From (2-36), (2-40), (2-44), and (2-49) we observe that k is a natural measure of the "favorableness" of the investment opportunities. This is because k is a maximum function determined by (the one-period utility function and) the distribution function F; moreover, F is reflected in the solution only through k, and $f(x)$ is increasing in k. Let us examine the effect of k on the marginal propensities to consume out of capital, $\partial c/\partial x$, and out of non-capital income, $\partial c/\partial y$, where $\partial c/\partial y = [1/(r-1)]\partial c/\partial x$. (By the linearity of the consumption function mentioned earlier, these propensities are constant over all feasible

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[1] Ibid., p. 134.

x and y.) From (2-34), we find that the propensity to consume is decreasing in k in the case of Model I. This phenomenon can at least in part be attributed to the fact that the utility function is bounded from below but not from above; the loss from postponement of current consumption is small compared to the gain from the much higher rate of consumption thereby made possible later. In Model II, where the utility function has an upper bound but no lower bound, the opposite is true. Here, the optimal amount of present consumption is increasing in k, which seems much more plausible from an intuitive standpoint.

In Model III, we observe from (2-42) the curious phenomenon that the optimal consumption strategy is independent of the investment opportunities in every respect. While the marginal propensity to consume is independent of k in Model IV also, the level of consumption is in this case an increasing function of k as is apparent from (2-47). We recall that the utility function in Model III is unbounded while that in Model IV is bounded both from below and from above. Thus, the class of utility functions we have examined implies an exceptionally rich pattern of consumption behaviour with respect to the "favorableness" of the investment opportunities.

2.7 THE BEHAVIOUR OF CAPITAL

We shall now examine the behaviour of capital implied by the optimal investment and consumption strategies of the different models. According to one school, capital growth is said to exist whenever

$$(2\text{-}58) \quad E[x_{j+1}] > x_j \qquad\qquad j = 1, 2, \ldots$$

that is, capital growth is defined as expected growth.[1] We shall reject this measure since x_j may under this definition, as $j \to \infty$, approach a value less than x_1 with a probability which tends to 1. We shall instead define growth as asymptotic growth, that is, capital growth is said to exist if

$$(2\text{-}59) \quad \lim_{j\to\infty} \Pr\{x_j > x_1\} = 1$$

When the > sign is replaced by the ≥ sign, we shall say that we have capital non-decline. If there is statistical independence with respect to j, (2-58) is implied by (2-59) but the converse does not hold, as noted.

Model IV will be considered first. From (2-50) it is clear that non-decline of capital is always implied (in fact, the solution to the problem is contingent upon the condition that capital does not decrease, as pointed out earlier). It is readily seen that a sufficient, but not necessary, condition for growth is that there is a non-zero investment in at least one of the risky investment opportunities since in that case, by (2-50), $\Pr\{x_{j+1} > x_j\} > 0$, $j = 1,2,\ldots$. A necessary and sufficient condition for asymptotic capital growth is $\alpha r > 1$, which is readily verified by reference to (2-50) and the foregoing statement.

Let us now turn to Models I-III. Defining s_j and p by the identities

$$(2\text{-}60) \quad s_j \equiv x_j + Y$$

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[1]See, for example, Edmund Phelps, op. cit., p. 735.

$$c(x) = p(x + Y)$$

we see from (2-34), (2-38), and (2-42) that p is a constant between 0 and 1, generally a different number for each model. By (2-2) we now obtain

$$\text{(2-61)} \quad s_{j+1} = s_j(1-p)\left[\sum_{i=2}^{M} (\beta_i - r)v_i^* + r\right]$$

$$= s_j W \qquad j = 1, 2, \ldots$$

where W is clearly a random variable. By the condition (2-8), it follows that $W \geq 0$. Attaching the subscript n to W for the purpose of period identification, we note that since

$$\text{(2-62)} \quad s_j = s_1 \prod_{n=1}^{j-1} W_n$$

we obtain

$$\text{(2-63)} \quad s_j \geq 0 \text{ for all } j \text{ whenever } s_1 \geq 0$$

That is, when initial capital is at least equal to the negative of the present value of the non-capital income stream (-Y), it will never decrease below that value when the individual uses the optimal strategies with respect to investment and consumption. Moreover, since $\Pr\{W > 0\} = 1$ in Models II and III by Corollary 1, it follows that

$$\text{(2-64)} \quad s_j > 0 \text{ whenever } s_1 > 0 \text{ for all finite } j$$

in Models II and III.

From (2-62) we also observe that

$$s_j = 0 \text{ whenever } s_k = 0 \text{ for all } j > k$$

Consequently, $x = -Y$ is a trapping state which, once entered, cannot be left. In this state, the optimal strategies in each case call for zero consumption, no productive investments, the borrowing of Y, and the payment of non-capital income y as interest on the debt. In Model I, whenever $\Pr\{W = 0\} > 0$, this state may clearly be reached after only one period, regardless of the initial capital position. In Models II and III, on the other hand, it follows from (2-64) that the trapping state will never be reached in a finite number of time periods if initial capital is greater than $-Y$.

(2-62) may be written

$$(2\text{-}65) \quad s_j = s_1 e^{\sum_{n=1}^{j-1} \log W_n}$$

The random variable $\sum_{n=1}^{j-1} \log W_n$ is by the Central Limit Theorem asymptotically normally distributed; its mean is $(j-1)E[\log W]$. By the law of large numbers,

$$(2\text{-}66) \quad \frac{\sum_{n=1}^{j-1} \log W_n}{j-1} \rightarrow E[\log W] \qquad \text{as } j \rightarrow \infty$$

Thus, since $s_j > s_1$ if and only if $x_j > x_1$, it is necessary and sufficient for capital growth to exist that

$$(2\text{-}67) \quad E[\log W] > 0$$

It is clear that μ given by

$$\mu = e^{E[\log W]}$$

may be interpreted as the mean growth rate of capital. By (2-61), we obtain

$$\text{(2-68)} \quad E[\log W] = \log(1-p) + E[\log\{\sum_{i=2}^{M} (\beta_i - r)v_i^* + r\}]$$

For Model III, this becomes by (2-44)

$$E[\log W] = \log \alpha + \max_{\substack{v_i \geq 0 \\ \forall i \notin S}} E[\log\{\sum_{i=2}^{M} (\beta_i - r)v_i + r\}]$$

subject to (2-8)

Thus, a person whose one-period utility function of consumption is logarithmic will always invest the capital available after the allotment to current consumption so as to maximize the mean growth rate of capital plus the present value of the non-capital income stream.

2.8 PROPERTIES OF THE OPTIMAL BORROWING AND LENDING STRATEGIES

The optimal amount to lend is given by $z_1(x)$; when this amount is negative, borrowing is called for. Denoting $\sum_{i=2}^{M} v_i^*$ by v^*, we obtain from (2-35), (2-39), and (2-43)

$$\text{(2-69)} \quad z_1(x) = (1-p)(1-v^*)x - Y[p(1-v^*) + v^*]$$

in Models I-III

and from (2-48)

$$\text{(2-70)} \quad z_1(x) = \frac{x}{r} - \frac{y}{r} + \frac{\log(-\alpha kr) - rv^*}{\gamma(r-1)}$$

in Model IV.

In each case, we find that lending is linear in wealth. Turning first to Model IV, we observe that lending is always increasing in x. Thus, when an individual in this model becomes sufficiently wealthy, he will always become a lender. At the other extreme, when x is at the lower boundary point of the solution set, he will generally be a borrower, though not necessarily, since $z_1(x)$ evaluated at $x = -Y + [r/(\gamma(r-1)^2)]\log(-\alpha kr)$ gives

$$-Y + \frac{r\log(-\alpha kr)}{\gamma(r-1)^2} - \frac{rv^*}{\gamma(r-1)}$$

which may be either negative or positive.

Turning to Models I-III, we find that borrowing always takes place at the lower end of the wealth scale; (2-69) evaluated at $x = -Y$ gives $-Y < 0$ as the optimal amount to lend. From (2-69) we also find that $z_1(x)$ is increasing in x if and only if $1 - v^* > 0$ since 1-p is always positive. As a result, the models always call for borrowing at least when the individual is poor; whenever $1 - v^* > 0$, they also always call for lending when he is sufficiently rich.

As remarked in 2.7, all non-capital income is allocated to the payment of interest once the trapping state is reached. Taken in conjunction with (2-63), this exemplifies the fact that when the individual behaves optimally, he behaves <u>as if</u> the following law were in effect: Your economic affairs must be so arranged that the probability is zero that your net worth at the end of any period will be less than the negative of the present value of your (certain) non-capital income stream. Needless to say, our solution is predicated on the existence of creditors who are willing to lend on this basis. This supposition is not as far-fetched as it may seem at first glance; many finance companies today lend large amounts on the basis of individuals' promised non-capital income streams. Moreover, as long as there is a market for the exchange of such loans, there is really no need for the principal ever to be collected as long as the interest is always paid in full when the loan reaches its maximum limit Y. Thus, even the trapping state itself is not entirely implausible, in this sense at least, from the standpoint of the real world.

2.8.1 The Existence of a Market Interest Rate

While this study is confined to an examination of the behaviour of individuals in a given but risky environment, it might be well to divert from this objective for a moment and consider how the interest rate $r-1$ might be determined in an economy where each individual behaves normatively in such a fashion that his behaviour is described by one of the models in this study.

By Lemmas 1 and 2, we know that v_i^* is unique in each of the models for each i. Allowing r to vary, we therefore have that v_i^* must be a monotonic function of r--it is readily determined that v_i^* is decreasing in r. Thus, as r increases, $1-v^*$ increases; consequently, lending, as a proportion of invested capital, is increased.

Differentiating (2-69) with respect to r we obtain

$$\frac{\partial z_1(x)}{\partial r} = -p'(r)(1 - v^*)x - (1 - p)v^{*'}(r)x$$

$$- Y'(r)[p(1 - v^*) + v^*]$$

$$- Y[p'(r)(1 - v^*) + v^{*'}(r)(1 - p)]$$

$$= (x + Y)[-p'(r)(1 - v^*) - v^{*'}(r)(1 - p)]$$

$$- Y'(r)[p + v^*(1 - p)]$$

$$> 0 \qquad \text{if } 0 \leq v^* \leq 1,\ p'(r) \leq 0$$

since $p > 0$, $v^{*'}(r) < 0$, and $Y'(r) < 0$. Since $p'(r) \leq 0$ in Models I and III and $0 \leq v^* \leq 1$ when short sales are not possible, total lending is always increasing in r in these two models when short sales are barred.

Now consider an economy in which individuals obey either Model I or Model III and which bars short sales. Assume first that the initial interest rate $r - 1 > 0$ causes the supply of loans to exceed the demand for loans. A reduction of the interest rate induces everyone, as we have seen, to lend less (or to borrow more). As $r \to 1$ from above, $Y \to \infty$ which implies that $z_1(x) \to -\infty$, that is, everyone's net borrowing tends to infinity, which is readily seen by reference to (2-69). By the monotonicity of $z_1(x)$ in r, the equilibrium rate of interest must therefore be positive.

Now assume that the initial interest rate causes the demand for loans to exceed the supply. By increasing the interest rate, equality between the supply and demand of funds will be achieved at some finite rate since $z_1(x) \to (1-p)x$ as $r \to \infty$, provided that total wealth is (substantially) positive because only then is it possible for each individual's consumption to be non-negative. Thus, in at least the case discussed, a unique, positive equilibrium rate of interest exists at each decision point even if the utility function, the wealth, the non-capital income stream, and the probability beliefs of each individual are different from everyone else's, as long as the combined wealth is (substantially) positive. By the stochastic nature of investment returns, the equilibrium rate would of course change in each period. This is also true in the real world. To avoid the sharp fluctuations that a pure market rate would exhibit, governments generally fix it artificially between narrow limits--this is the reason why the existence of a given fixed rate of interest was postulated in this study.

2.8.2 Different Rates for Borrowing and Lending

We shall now consider the case when the lending rate differs from the borrowing rate as is usually the case in the real world. We observed in 2.8.1 that the v_i^*, and therefore v^*, are decreasing functions of r; let us write the latter as $v^*(r)$. Thus,

$$(2\text{-}71) \quad v^*(r_L) > v^*(r_B) \qquad \text{whenever } r_L < r_B.$$

Consider first Models I-III when non-capital income $y = 0$. In that case, it is apparent from (2-69) that when the individual is not in the trapping state, he either always borrows, always lends, or does neither, depending on whether $1 - v^*(r)$ is negative, positive, or zero. It follows that if he should lend at the lending rate, he should also do so at the borrowing rate, if he could. Similarly, if he should borrow at the borrowing rate he should also do so at the lower lending rate, were it possible. Thus, in these cases, the solution is clear. The only remaining possibility is that the lending rate calls for borrowing and that the borrowing rate calls for lending. By the monotonicity of $v^*(r)$, the solution in this case is to neither borrow nor lend.

Turning to Model IV, we observe from (2-70) that when $y = 0$, $z_1(x)$ is non-negative for all feasible capital positions whenever $v^*(r) \leq \log(-\alpha kr)/r$. If this condition holds when $r = r_L$, it is apparent that it need not also hold for $r = r_B$. Consequently, no "simple" solution appears to exist in the case of Model IV.

When $y > 0$, the problem becomes more complicated. However, if in Models I-III $1-v^*(r_B)$ is negative, then positive lending would never occur since use of the lending rate would also always call for borrowing by (2-71). Thus, in this case, the solutions are as stated with r replaced by r_B.

2.9 PROPERTIES OF THE OPTIMAL INVESTMENT STRATEGIES

The properties exhibited by the optimal investment strategies are in a sense the most interesting. Turning first to Model IV, we note that the portfolio of productive investments is constant, both in mix and amount, at all levels of wealth. The optimal portfolio is also independent of the non-capital income stream $(y_1, y_2, \ldots)$, and the level of impatience $1-\alpha$ possessed by the individual, as shown by (2-48) and (2-49).

Similarly, we find in Models I-III that since for all $i,k > 1$, $z_i(x)/z_k(x) = v_i^*/v_k^*$ which is a constant, the <u>mix</u> of risky investments is independent of wealth, non-capital income, and impatience to spend. However, the <u>size</u> of the total investment commitment in each period is clearly increasing in x and in y. We also note that when $y = 0$, the ratio that the risky portfolio $\sum_{i=2}^{M} z_i(x)$ bears to the total portfolio $\sum_{i=1}^{M} z_i(x)$ is independent of wealth in each model.

In summary, then, we have the surprising result that the optimal mix of risky (productive) investments in each of Models I-IV is independent of the individual's wealth, non-capital income stream, and rate of impatience to consume; the optimal mix depends in each case only on the probability distributions of the returns, the interest rate, and the individual's one-period utility function of consumption.

As to the composition of the optimal portfolio, we note from Corollary 4 that when the returns from the various risky opportunities are independently distributed and the one-period utility function of consumption has no lower bound, the portfolio will be long with respect to all opportunities i such that $E[\beta_i] > r$ and short with respect to all opportunities $i \in S$ such that $E[\beta_i] < r$; all other opportunities will be excluded.

2.10 <u>GENERALIZATIONS</u>

We shall now drop the temporary restrictions adopted at the end of 2.1 concerning the non-capital income stream and the time dependence of the distribution functions of return F_j. Before doing so, we shall briefly discuss the decision problem under study when the planning horizon is finite.

2.10.1 Finite Horizon

While we have chosen to examine the preceding decision problem for the case when the utility function is defined over an infinite future, the finite horizon problem, as suggested earlier, is just as amenable to solution. In fact, the initial approximation, $f_1(x)$, in (2-17) was chosen in such a way that the successive approximations would give the solutions to the finite horizon problem. Thus, the solution to Models I and II when the horizon is N periods away is given by (2-26)-(2-28).

2.10.2 Non-Constant Non-Capital Income Stream

We shall now consider the case when the non-capital income in period j, y_j, is not necessarily equal to y as we have assumed so far. It is now necessary to reinstate the subscript j for variables x, c, and z_i as well as for Y and f. It is easy to show that the solutions to Models I-IV remain unchanged except that the subscript j must be appended as indicated. Condition (2-67) is still necessary and sufficient for capital growth in Models I-III; since $x_j > x_1$ if and only if $s_j > s_1 + Y_j - Y_1$, capital growth exists by (2-66) if and only if, for large j

$$e^{(j-1)E[\log W]} > \frac{x_1 + Y_j}{x_1 + Y_1}$$

or

$$E[\log W] > \lim_{j\to\infty} \frac{1}{j-1} \log\left[\frac{x_1 + Y_j}{x_1 + Y_1}\right] = 0$$

since Y_j is bounded.

2.10.3 Time-Dependent Probability Distributions

The assumption that the transformation β_{ij} is identically distributed in each period j for any given i will now be dropped. This also enables us to consider automatically the case when the number of investment opportunities M_j varies from period to period.

It is again easily shown, by the method of successive approximations applied as one would solve the finite horizon problem, that the structure of the solutions to Models I-IV is not affected by this change. The subscript j must of course appear for β_{ij} as well as for x, c, z_i, Y, and f; it must also be attached to the constant k in (2-36), (2-40), (2-44), and (2-49) as well as to the variables v_i. This, in turn, affects the convergence condition in Model I and condition (2-50) in Model IV. Before describing the latter changes, let us give the solution to Model II, which exists for $x_j \geq -Y_j$, when the probability distributions of return are time-dependent and the non-capital income stream is non-constant:

$$\text{(2-72)} \quad f_j(x_j) = -\left[1 + [\alpha(-k_j)]^{\frac{1}{\gamma+1}} + [\alpha^2(-k_j)(-k_{j+1})]^{\frac{1}{\gamma+1}} + \dots\right]^{\gamma+1} (x_j + Y_j)^{-\gamma}$$

$$(2\text{-}73)\quad c_j(x_j) = \frac{1}{1 + [\alpha(-k_j)]^{\frac{1}{\gamma+1}} + [\alpha^2(-k_j)(-k_{j+1})]^{\frac{1}{\gamma+1}} + \ldots}(x_j + Y_j)$$

$$z_{ij}(x_j) = (x_j + Y_j - c_j)v^*_{ij} \qquad i = 2, \ldots, M_j$$

$$(2\text{-}74)\quad z_{1j}(x_j) = x_j - c_j - \sum_{i=2}^{M_j} z_{ij}(x_j)$$

where k_j and the v^*_{ij} are given by

$$(2\text{-}75)\quad k_j = E\left[-\left\{\sum_{i=2}^{M_j}(\beta_{ij}-r)v^*_{ij} + r\right\}^{-\gamma}\right]$$

$$= \max_{\substack{v_{ij}\geq 0\\ \forall i \notin S_j}} E\left[-\left\{\sum_{i=2}^{M_j}(\beta_{ij}-r)v_{ij} + r\right\}^{-\gamma}\right]$$

subject to

$$\Pr\left\{\sum_{i=2}^{M_j}(\beta_{ij}-r)v_{ij} + r \geq 0\right\} = 1$$

The convergence condition $\alpha k < 1$ in Model I must now be replaced by the requirement

$$(2\text{-}76)\quad \alpha k_j < 1 \quad \text{for all } j \geq N$$

where N is a positive integer. Similarly, the condition (2-50) in Model IV now becomes

(2-77) $\log(-\alpha k_j r) + b(\bar{v}_j^*) \geq 0$ all j

and the solution exists in this model for x_j greater than or equal to

$$\operatorname*{Max}_j \left\{ -Y_j + \frac{r}{\gamma(r-1)^2} \log(-\alpha k_j r) \right\}$$

The necessary and sufficient condition for capital growth (2-67) now reduces, in Models I-III, to

(2-78) $E[\log W_j] > 0$ all $j \geq N$

where N is a positive integer.

It should be noted that in order to find the optimal mix of risky investments at a given decision point, it is only necessary to know the probability distributions of return for the <u>current</u> period. This is so in all of the models. However, in order to determine the optimal amount to consume in a given period, the distributions must be known for the entire future, except in Model III. In this model, the optimal consumption strategy is independent of the return distributions, as shown by (2-42). Consequently, the only knowledge about that part of the future which is more than one period ahead needed to maximize utility over time in Model III, besides the utility function itself, is a knowledge of the non-capital income stream and of the interest rate. In essence, then, when the one-period utility function of consumption is logarithmic, the sequential decision problem reduces to a sequence of one-period decision problems, one of which is to be solved at each decision point.

2.11 IMPLICATIONS WITH RESPECT TO THE THEORY OF THE FIRM

We have shown that the utility functions of Models I-IV imply that the optimal mix of risky investments in any period is independent of wealth, non-capital income, and impatience to spend. We shall now show that it would be reasonable to conjecture that, when the utility function is of the form (1-5), these functions (i.e., those of Models I-IV) are the <u>only</u> <u>ones</u> for which this property of the optimal investment strategies holds.

To show this, assume that a solution to (2-6) exists and that the optimal investment strategy calls for the mix $(v^*_{2j}, v^*_{3j}, \ldots, v^*_{M_j j})$ regardless of the values of x_j, Y_j, and α, where at least two of the $v^*_{ij} \neq 0$ for each j. Letting $v^*_j \equiv \sum_{i=2}^{M_j} v^*_{ij}$, v^*_{ij}/v^*_j then denotes the proportion of the funds devoted to productive (risky) investments which is allocated to opportunity i in period j.

Differentiating (2-6) with respect to z_{ij}

$$\text{(2-79)} \quad \frac{\partial f_j}{\partial z_{ij}} = \alpha E\left[f'_{j+1}\left(\sum_{i=2}^{M_j} (\beta_{ij}-r) z_{ij} + r(x_j - c_j) + y_j \right)(\beta_{ij}-r) \right]$$

all j, i = 2, ..., M_j

The maximizing z_{ij} must now be of the form

$$\text{(2-80)} \quad z_{ij}(x_j) = v^*_{ij} q_j \qquad \text{all } i, j$$

where q_j is some function of x_j, c_j, and Y_j. In the case of an interior maximum (which cannot be ruled out)

(2-81) $\frac{\partial f_j}{\partial z_{ij}} = 0$ all i, j

Given that f_j is non-decomposable, i.e., it cannot be written as the sum or product of two or more functions, it appears possible for (2-81) to hold for all x_j and Y_j when the maximizing z_{ij} are of the form (2-80) only if $c_j = x_j + y_j/r$ and q_j is constant or if the β_{ij} are identically distributed for all i such that $v^*_{ij} \neq 0$ given j. Since the latter possibility is ruled out by even the mildest requirement of generality with respect to the investment returns, it remains to consider whether it is feasible for c_j to be equal to $x_j + y_j/r$ when q_j is constant. We obtain when this is the case

$$\frac{\partial f_j}{\partial c_j} = u'(x_j + \frac{y_j}{r}) - \alpha r E\left[f'_{j+1}(\sum_{i=2}^{M_j}(\beta_{ij}-r)v^*_{ij}q_j)\right]$$

$$= u'(x_j + \frac{y_j}{r}) + \bar{K}_j \qquad (\bar{K}_j \text{ constant})$$

so that $c_j = x_j + y_j/r$ with q_j constant is therefore not the optimal consumption strategy for all values of x_j and Y_j. Thus, when f_j is non-decomposable, it would appear that an investment strategy such that the mix of risky investments is invariant for all x_j, α, and Y_j is never optimal.

Let us therefore consider the situation when f_j is decomposable. In this case, one of the four Cauchy equations must hold. That is, either

(2-82) $f_j(x + y) = g_j(x) + h_j(y)$

or

$$f_j(x + y) = g_j(x)h_j(y) \tag{2-83}$$

or

$$f_j(xy) = g_j(x) + h_j(y) \tag{2-84}$$

or

$$f_j(xy) = g_j(x)h_j(x) \tag{2-85}$$

for all j. Their non-trivial solutions are given by[1]

$$f_j(t) = a_j t + b_j + d_j, \; g_j(t) = a_j t + b_j, \; h_j(t) = a_j t + d_j \tag{2-86}$$

$$f_j(t) = a_j b_j e^{\gamma_j t}, \; g_j(t) = a_j e^{\gamma_j t}, \; h_j(t) = b_j e^{\gamma_j t} \tag{2-87}$$

$$f_j(t) = a_j \log(b_j d_j t), \; g_j(t) = a_j \log(b_j t), \; h_j(t) = a_j \log(d_j t) \tag{2-88}$$

$$f_j(t) = a_j b_j t^{\gamma_j}, \; g_j(t) = a_j t^{\gamma_j}, \; h_j(t) = b_j t^{\gamma_j} \tag{2-89}$$

respectively, where a_j, b_j, d_j, and γ_j are constants.

Consider (2-89) first. (2-6) now becomes

$$a_j b_j x_j^{\gamma_j} = \max\left\{ u(c_j) + \alpha a_{j+1}\left(x_j + \frac{y_j}{r} - c_j\right)^{\gamma_{j+1}} b_{j+1} E\left[\left\{ \sum_{i=2}^{M_j} (\beta_{ij} - r) \frac{z_{ij}}{x_j + \frac{y_j}{r} - c_j} + r \right\}^{\gamma_{j+1}}\right] \right\} \tag{2-90}$$

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[1] See J. Aczél, Vorlesungen . . ., pp. 116-118.

But when we maximize with respect to c_j and z_{ij}, we know from Theorem 2 that the last factor is equal to k_j as given by (2-36) or (2-40) and also that the optimal investment mix is invariant for all x_j, Y_j, and α. Thus, we are left with an equation of the form

$$a_j b_j x_j^{\gamma_j} = \max_{0 \le c_j \le x_j + Y_j} \left\{ u(c_j) + a_{j+1} b_{j+1} \alpha k_j \left(x_j - c_j + \frac{y_j}{r}\right)^{\gamma_{j+1}} \right\}$$

Consequently, $u(c_j)$ must satisfy the equation

(2-91) $$a_j b_j c_j^{\gamma_j} = u(c_j) + a_{j+1} b_{j+1} \alpha k_j (y_j/r)^{\gamma_{j+1}}$$

so that

$$u(c) = Kc^{\gamma} + a \qquad (K,\ a,\ \gamma \text{ constants})$$

which leaves as the only possible utility functions

(2-15) $u(c) = c^{\gamma}$ $\qquad 0 < \gamma < 1$

(2-16) $u(c) = -c^{-\gamma}$ $\qquad \gamma > 0$

Similarly, from (2-87) and (2-88) we obtain

(2-45) $u(c) = -e^{-\gamma c}$ $\qquad \gamma > 0$

(2-92) $u(c) = \log c$

while no utility function exists in the case of (2-86) due to the linearity of that function.

We shall now summarize the main result obtained in this study in the following theorem:

Theorem 5. Given the following:

1) An individual whose economic objective is the maximization of expected utility from consumption over time and whose preferences are representable by

2) A utility function $U = \sum_{j=1}^{\infty} \alpha^{j-1} u(c_j)$, $0 < \alpha < 1$, defined for all $c_j \geq 0$, where c_j is the amount consumed in period j, such that $u(c_j)$ is strictly concave, twice differentiable, and monotone increasing over its entire domain.

3) Interest rate $r-1 > 0$ at which funds may be both borrowed and lent.

4) A non-capital income stream $(y_j, y_{j+1}, \ldots)$ where y_j is the instalment received with certainty at the end of period j. Its present value at the jth decision point, on the basis of interest rate r-1, is Y_j.

5) Wealth x_j at the jth decision point where $x_1 \geq -Y_1$.

6) The opportunity to lend unrestricted amounts and to borrow up to amount Y_j on the security of the non-capital income stream alone. Additional funds may be borrowed to the extent that principal and interest can be repaid with certainty in one period.

7) The opportunity to invest at each decision point j in ventures which are subject to risk. For each venture, the return is realized in cash at the end of the period; returns to scale are constant. If β_{ij} is the transformation in period j of a unit of capital invested in opportunity i, then $0 \leq \beta_{ij} < \infty$ for all i and j. The distribution functions F_j where

$$F_j(x_2, x_3, \ldots, x_{M_j}) \equiv \Pr\{\beta_{2j} \le x_2,\ \beta_{3j} \le x_3, \ldots, \beta_{M_j j} \le x_{M_j}\}$$

are assumed to be known and independent for all j. Moreover, $\Pr\{\sum_{i=2}^{M_j} (\beta_{ij} - r)\theta_{ij} < 0\} > 0$ for all j where the θ_{ij} are finite numbers such that $\theta_{ij} \ge 0$ for all opportunities i which cannot be sold short in period j and $\theta_{ij} \ne 0$ for at least one i.

Then the optimal mix of risky investments in each period j is independent of the wealth x_j, the non-capital income stream $(y_j, y_{j+1}, \ldots)$, and the impatience rate $1-\alpha$ if the one-period utility function u(c) is such that either the risk aversion index q(c) given by

$$(2\text{-}54) \quad q(c) = -\frac{u''(c)}{u'(c)}$$

or the proportional index $q^*(c)$ given by

$$(2\text{-}93) \quad q^*(c) = -\frac{u''(c)c}{u'(c)}$$

is a positive constant, that is, u(c) is one of the functions (2-15), (2-16), (2-92), or (2-45). Alternatively, the optimal mix of risky investments in any period depends only on the set of distribution functions $\{F_j\}$, the interest rate r-1, and the (risk aversion) index q or q^*.

The implications of this result are particularly significant in respect to the theory of the firm as we shall now demonstrate.

2.11.1 Bases for the Formation of Firms

The following consequence is immediate:

Corollary 5. Given the antecedents of Theorem 5 and a collection of individuals whose risk aversion indices q or q^* are positive constants, there is a basis for the formation of firms, one for each non-identical pair $(q,\{F_j\})$ and $(q^*,\{F_j\})$, such that individuals with the same risk aversion index and the same probability beliefs may delegate the choice and mix of productive (risky) investments to the same firm regardless of their wealth, non-capital income, and impatience to spend.

By the words "may delegate" we mean that the individual would be indifferent between making the risky investments himself and turning the total amount allocated to risky investments over to a firm for investment. The firm $(q^*,\{F_j\})$ will be said to be compatible with individuals whose risk aversion indices are q^* and whose probability beliefs are given by $\{F_j\}$.

This result may be interpreted in two different ways. From a normative point of view, it indicates that there is a rational reason for economic cooperation (or, more accurately in our framework, no rational reason for non-cooperation) among individuals with different goals and in different economic circumstances. Descriptively, the corollary implies that the economic cooperation we observe among unlike individuals in the real world is consistent with, and may possibly have arisen as a result of, each individual's (selfish) desire to maximize expected utility from consumption over time.

2.11.2 The Firm's Objective and Its Optimal Capital Structure

We shall now continue on the path which has just been broken. The next implication which we shall state formally is

Theorem 6. Given the antecedents of Theorem 5 and a firm in the sense of Corollary 5, the objective of the firm may be stated as: In each period j, invest proportion v_{ij}^*/v_j^* of all capital in activity i, $i = 2, \ldots, M_j$, where $v_j^* = \sum_{i=2}^{M_j} v_{ij}^*$ and the v_{ij}^* are the values of v_{ij} which maximize

$$(2\text{-}94) \qquad E[u(\sum_{i=2}^{M_j}(\beta_{ij}-r)v_{ij} + r)]$$

subject to (2-8) and (2-9) whenever $-cu''(c)/u'(c)$ is a positive constant, or which maximize

$$(2\text{-}95) \qquad E[u(\sum_{i=2}^{M_j}(\beta_{ij}-r)v_{ij})]$$

subject to (2-9) whenever $-u''(c)/u'(c)$ is a positive constant, assuming, in each case, that $v_j^* > 0$. Moreover, when the firm has unlimited liability, the optimal capital structure (the ratio between debt and equity capital) of the firm is arbitrary.

Proof: The first part of the theorem follows immediately from (2-35), (2-39), (2-43), and (2-48). Turning to the second part, let the individual's optimal investment strategy call for allocating, in a given period, amount a_1 to risky investments and a_2 to lending (borrowing when a_2 is negative). Let the (final) debt-equity ratio of the firm at the decision point in question be $\theta \geq 0$. Then by investing $a_1/(1+\theta)$ of equity capital in the firm, the individual obtains the benefit of $(1 + \theta)a_1/(1+\theta) = a_1$ invested in the risky investments. If the individual lends the difference $a_1 - a_1/(1+\theta)$, his lending becomes $a_2 + a_1 - a_1/(1+\theta)$. But his "share" of the debt of the firm is $\theta a_1/(1+\theta)$; his "net" lending is therefore $a_2 + a_1 - a_1/(1+\theta) - \theta a_1/(1+\theta) = a_2$. Thus, θ may be chosen arbitrarily.

Thus, starting with a collection of heterogeneous individuals, each of whom is bent on maximizing (his own) utility from consumption over time, we have not only found that there exists a basis for the formation of firms by <u>sub-collections</u> of individuals (each sub-collection in turn possessing significant heterogeneity), but that each such firm has a well-defined (unique) objective function and that its capital structure (debt-equity ratio) is unimportant. The firm's objective function may be said to call for "profit maximization" where the precise meaning of this term <u>under risk and with respect to time</u> has been induced from the most primitive of inputs: the preferences, the economic circumstances, and the perception of opportunities of its owners. Note that the maximization of (2-94) or (2-95) may be viewed as the short-run objective of the firm but that, performed repeatedly, the short-run maximization process also yields the long-run objective.

Concerning the optimal capital structure of the firm, Theorem 6 has an interesting relation to Proposition I of Modigliani and Miller: "The market value of any firm is independent of its capital structure and is given by capitalizing its expected return at the rate σ_k appropriate to its class."[1] To the extent that this proposition may be interpreted to mean that the optimal capital structure of the firm is arbitrary, Theorem 6 does, of course, support the proposition.

Perhaps the chief practical significance of Theorem 6 lies in the fact that it is often easier for the firm to borrow money than for the individual to do so, especially when his net worth is negative. As indicated in the proof, individuals have the opportunity to shift, without loss of utility, a large portion of their borrowing requirements to the firm. Whenever $a_2 < 0$ and θ is such that $a_1 - a_1/(1+\theta) > |a_2|$, we find that the individual should, instead of borrowing on his own account, become a lender. Clearly his lending might take the form of a position in the bonds of his own firm!

When $y_j = 0$ for all j, the ratio between the financial portfolio and the productive portfolio is constant for all x_j since $z_{1j}(x_j)/\sum_{i=2}^{M_j} z_{ij}(x_j) = (1-v_j^*)/v_j^*$ which is a constant. Thus, the owners may in this case delegate <u>all</u> of their borrowing (or lending) requirements to the firm.

Note also that when $y_j = 0$ for all j and $u(c) = \log c$, the function (2-94) to be maximized by the firm coincides with that proposed by Breiman and by Brown; by maximizing the expected logarithm of capital in each period, the long-run capital position of the firm

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[1] Franco Modigliani and Mervin Miller, "The Cost of Capital, Corporation Finance and the Theory of Investment," <u>American Economic Review</u>, June 1958, p. 268.

will be greater than under any other policy with a probability approaching 1.[1] We have thus found (rational) stockholders or partners who would subscribe to the firm investment strategy suggested by Breiman and by Brown.

2.11.3 The Debt of the Firm: Limited Liability

When the legal status of the firm is such that it has limited liability (e.g., it is a corporation), the amount the firm may borrow is clearly not arbitrarily large since lenders expect both interest and the repayment of principal with probability 1(at least in our model). However, since they have first claim against the firm's assets, there is a limit up to which they should not hesitate to lend. Formally we obtain

Theorem 7. Given a firm in the sense of Theorem 6 but with limited liability, the optimal capital structure of the firm is arbitrary except that there is an upper limit on the firm's feasible debt-equity ratio imposed by the limit attached to its liability. The upper limit on the debt-equity ratio in period j is given by

$$\frac{1 + b(\bar{v}_j^*)/v_j^*}{r - 1 - b(\bar{v}_j^*)/v_j^*} \tag{2-96}$$

where $b(\bar{v}_j^*)$ is the greatest lower bound on b such that

$$\Pr\{\sum_{i=2}^{M_j} (\beta_i - 1)v_{ij}^* < b\} > 0$$

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[1] Leo Breiman, "Investment Policies for Expanding Businesses Optimal in a Long-Run Sense," Naval Research Logistics Quarterly, December 1960; George Brown, Unpublished Notes, (1964).

and $\bar{v}_j^* \equiv (v_{2j}^*, \ldots, v_{M_j j})$.

<u>Proof</u>: Let θ_j be the firm's debt-equity ratio and x_j its equity capital at the jth decision point. Then the greatest lower bound on the firm's assets at the end of period j is

$$(2\text{-}97) \quad x_j(\theta_j+1)\ [1 + b(\bar{v}_j^*)/v_j^*]$$

by Theorem 6 while the creditors are owed $x_j\theta_j r$. Thus, it is necessary for (2-97) to be at least as large as $x_j\theta_j r$ in order for the creditors to receive their due with probability 1. Solving for θ_j we obtain (2-96). It follows from (2-8) and the "no-easy-money condition" in 2.1 that $-1 \leq b(\bar{v}_j^*)/v_j^* < r - 1$ since the firm can only exist if $v_{ij}^* \neq 0$ for at least one i and we find that $b(\bar{v}_j^*)/v_j^* < r - 1$ for all such $\bar{v}_j^*$.

It is readily seen that when $b(\bar{v}_j^*)/v_j^*$ is close to its upper limit, the maximum debt-equity ratio will be relatively large, and vice-versa. This may in part explain why public utilities, for whom losses, if any, are typically small, are often found to have a higher debt-equity ratio than for example small electronics companies, for whom large losses are much more probable. This is because almost any selection process in which each non-degenerate interval on the feasible debt-equity ratio scale of the firm has a positive probability of being chosen would produce this result. (Since the optimal debt-equity ratio is arbitrary, any selection process which will choose a feasible ratio with probability 1 would of course also be optimal.)

CHAPTER III

APPLICATIONS AND EXAMPLES

In this chapter, we shall illustrate the results obtained in Chapter II by means of examples. In addition, we shall discuss briefly some of the applications to which the model studied there lends itself.

3.1 INDIVIDUAL DECISION-MAKING

The central core around which the model in this study was developed is the individual, faced with the economic decisions he must unavoidably make, whether consciously or subconsciously. Let us now compute what his optimal decisions would be for certain utility functions and under a given set of opportunities. For illustration, let the financial and productive opportunities be as follows in each period.

$$\beta_1 = r = 1.05$$

$$\beta_2 = \begin{cases} .95 \text{ with probability } .5 \\ 1.20 \text{ with probability } .5 \end{cases}$$

$$\beta_3 = \begin{cases} .75 \text{ with probability } .6 \\ 2.00 \text{ with probability } .4 \end{cases} \tag{3-1}$$

$$\beta_4 = \begin{cases} .60 \text{ with probability } .3 \\ 1.20 \text{ with probability } .7 \end{cases}$$

For simplicity, we shall assume that β_2, β_3, and β_4 are independently distributed and that each opportunity may be sold short. When the patience rate α is .88 and the non-capital income is \$10,000 in each period, the optimal consumption and investment decisions for the one-period utility functions $u(c) = \sqrt{c}$, $u(c) = -c^{-2}$, and $u(c) = \log c$, respectively, are given in Tables II, III, and IV. As the tables show, the person whose one-period utility function is $\sqrt{c}$ invests and

TABLE II

OPTIMAL ALLOCATION OF CAPITAL (AT EACH DECISION POINT) FOR SELECTED CAPITAL POSITIONS WHEN $u(c) = \sqrt{c}$, $\alpha = .88$, $y = \$10,000$ IN EACH PERIOD, AND r, β_2, β_3, β_4 ARE AS IN (3-1)

Capital Position	Optimal Allocation of Capital							
	Con- sumption	Investments						
			When Made by Individual Directly				When There is a Compatible Firm With Debt-Equity Ratio 3:1	
		Lending (Borrowing)	Productive Opportunities				Lending (Borrowing)	Equity Capital of Firm
x	c		Total	#2 (73.5%)	#3 (45.2%)	#4 (-18.7%)		
$-200,000	-	$ (200,000)	-	-	-	-	$(200,000)	-
-100,000	$ 4,000	(412,160)	$ 308,160	$ 226,560	$ 139,200	$-57,600	(181,040)	$ 77,040
- 50,000	6,000	(518,240)	462,240	339,840	208,800	-86,400	(171,560)	115,560
- 20,000	7,200	(581,888)	554,688	407,808	250,560	-103,680	(165,872)	138,672
- 10,000	7,600	(603,104)	585,504	430,464	264,480	-109,440	(163,976)	146,376
0	8,000	(624,320)	616,320	453,120	278,400	-115,200	(162,080)	154,080
10,000	8,400	(645,536)	647,136	475,776	292,320	-120,960	(160,184)	161,784
20,000	8,800	(666,752)	677,952	498,432	306,240	-126,720	(158,288)	169,488
50,000	10,000	(730,400)	770,400	566,400	348,000	-144,000	(152,600)	192,600
100,000	12,000	(836,480)	924,480	679,680	417,600	-172,800	(143,120)	231,120
200,000	16,000	(1,048,640)	1,232,640	906,240	556,800	-230,400	(124,160)	308,160
500,000	28,000	(1,685,120)	2,157,120	1,585,920	974,400	-403,200	(67,280)	539,280
1,000,000	48,000	(2,745,920)	3,697,920	2,718,720	1,670,400	-691,200	27,520	924,480

Maximum Loss in Each Period in Terms of $x - c + Y$: 71.1 %

Mean Growth Rate of $x + Y$: 1.6 %.

TABLE III

OPTIMAL ALLOCATION OF CAPITAL (AT EACH DECISION POINT) FOR SELECTED CAPITAL POSITIONS

When $u(c) = -c^{-2}$, $\alpha a = .88$, $y = \$10,000$ IN EACH PERIOD, AND r, β_2, β_3, β_4 ARE AS IN (3-1)

Capital Position x	Optimal Allocation of Capital							
	Con-sumption c	Investments						
		When Made by Individual Directly					When There is a Compatible Firm With Debt-Equity Ratio 3:1	
		Lending (Borrowing)	Productive Opportunities				Lending (Borrowing)	Equity Capital of Firm
			Total	#2 (89.1%)	#3 (34.3%)	#4 (-23.4%)		
$ -200,000	-	$ (200,000)	-	-	-	-	$ (200,000)	-
-100,000	$ 9,000	(167,240)	$ 58,240	$ 51,870	$ 20,020	$-13,650	(123,560)	$ 14,560
- 50,000	13,500	(150,860)	87,360	77,805	30,030	-20,475	(85,340)	21,840
- 20,000	16,200	(141,032)	104,832	93,366	36,036	-24,570	(62,408)	26,208
- 10,000	17,100	(137,756)	110,656	98,553	38,038	-25,935	(54,764)	27,664
0	18,000	(134,480)	116,480	103,740	40,040	-27,300	(47,120)	29,120
10,000	18,900	(131,204)	122,304	108,927	42,042	-28,665	(39,476)	30,576
20,000	19,800	(127,928)	128,128	114,114	44,044	-30,030	(31,832)	32,032
50,000	22,500	(118,100)	145,600	129,675	50,050	-34,125	(8,900)	36,400
100,000	27,000	(101,720)	174,720	155,610	60,060	-40,950	29,320	43,680
200,000	36,000	(68,960)	232,960	207,480	80,080	-54,600	105,760	58,240
500,000	63,000	29,320	407,680	363,090	140,140	-95,550	335,080	101,920
1,000,000	108,000	193,120	698,880	622,440	240,240	-163,800	717,280	174,720

Maximum Loss in Each Period in Terms of x - c + Y: 9.6%

Mean Growth Rate of x + Y: .2%.

TABLE IV

OPTIMAL ALLOCATION OF CAPITAL (AT EACH DECISION POINT) FOR SELECTED CAPITAL POSITIONS WHEN $u(c) = \log c$, $\alpha = .88$, $y = \$10,000$ IN EACH PERIOD, AND r, β_2, β_3, β_4 ARE AS IN (3-1)

Capital Position	Optimal Allocation of Capital							
		Investments						
		When Made by Individual Directly					When There is a Compatible Firm With Debt-Equity Ratio 3:1	
	Con-sumption	Lending (Borrowing)	Productive Opportunities				Lending (Borrowing)	Equity Capital of Firm
x	c		Total	#2 (84.0%)	#3 (37.8%)	#4 (-21.8%)		
$ -200,000	-	$ (200,000)	-	-	-	-	$ (200,000)	-
-100,000	$ 12,000	(277,440)	$ 165,440	$ 139,040	$ 62,480	$-36,080	(153,360)	$ 41,360
- 50,000	18,000	(316,160)	248,160	208,560	93,720	-54,120	(130,040)	62,040
- 20,000	21,600	(339,392)	297,792	250,272	112,464	-64,944	(116,048)	74,448
- 10,000	22,800	(347,136)	314,336	264,176	118,712	-68,552	(111,384)	78,584
0	24,000	(354,880)	330,880	278,080	124,960	-72,160	(106,720)	82,720
10,000	25,200	(362,624)	347,424	291,984	131,208	-75,768	(102,056)	86,856
20,000	26,400	(370,368)	363,968	305,888	137,456	-79,376	(97,392)	90,992
50,000	30,000	(393,600)	413,600	347,600	156,200	-90,200	(83,400)	103,400
100,000	36,000	(432,320)	496,320	417,120	187,440	-108,240	(60,080)	124,080
200,000	48,000	(509,760)	661,760	556,160	249,920	-144,320	(13,440)	165,440
500,000	84,000	(742,080)	1,158,080	973,280	437,360	-252,560	126,480	289,520
1,000,000	144,000	(1,129,280)	1,985,280	1,668,480	749,760	-432,960	359,680	496,320

Maximum Loss in Each Period in Terms of x - c + Y: 38 %

Mean Growth Rate of x + Y: .8 %.

borrows more and is able to achieve a higher growth rate than persons whose one-period utility functions are $-c^{-2}$ or log c, given the same opportunities. However, the price he pays is reflected in his consumption rate, which is the lowest of the three. The highest consumption rate is exhibited by the person whose utility function is logarithmic; note also that this person's investments and capital growth are greater than those associated with utility function $-c^{-2}$. It is apparent from the tables that the wealth of the individual in Table II ($u(c) = \sqrt{c}$) fluctuates the most and that of the individual in Table III ($u(c) = -c^{-2}$) the least. In the examples, only the person in Table III ($u(c) = -c^{-2}$) will ever be a lender.

Since no solution exists in the case of Model IV when $-\alpha kr < 1$, we have summarized in Table V the optimal capital allocation for the case when $u(c) = -e^{-.0001c}$, $\alpha = .99$, $r = 1.06$, and β_2 may assume each of the values .96 and 1.17 with probability .5 (one risky opportunity).

Table VI gives an idea of how the optimal consumption and investment decisions are affected by differences in the impatience rate. As patience increases, it is seen that current consumption decreases and investment increases. The effect of the size of the non-capital income stream is illustrated in Table VII. This table also demonstrates that individuals who have the same $x + Y$ $(= x + y/r - 1)$ (should) make the same decisions with respect to consumption and investments. Only their borrowing and lending differ - by the amount that the present values of their non-capital income streams differ. Recall that the mix of productive (risky) investments is unaffected,

TABLE V

OPTIMAL ALLOCATION OF CAPITAL (AT EACH DECISION POINT) FOR SELECTED CAPITAL POSITIONS WHEN $u(c) = -e^{-.0001c}$, $\alpha = .99$, $y = \$10,000$ IN EACH PERIOD, $r = 1.06$, AND β_2 MAY ASSUME EACH OF THE VALUES .96 AND 1.17 WITH PROBABILITY .5

Capital Position x	Con-sumption c	Optimal Allocation of Capital: Investments: When Made by Individual Directly: Lending (Borrowing)	Productive Opportunity	When There is a Compatible Firm With Debt-Equity Ratio 3:1: Lending (Borrowing)	Equity Capital of Firm
$ -22,986	-	$ (102,488)	$79,502	$ (42,862)	$19,876
-15,000	$ 452	(94,954)	79,502	(35,328)	19,876
-10,000	735	(90,237)	79,502	(30,611)	19,876
0	1,301	(80,803)	79,502	(21,177)	19,876
10,000	1,867	(71,369)	79,502	(11,743)	19,876
20,000	2,433	(61,935)	79,502	(2,309)	19,876
50,000	4,131	(33,633)	79,502	25,993	19,876
100,000	6,961	13,537	79,502	73,163	19,876
200,000	12,621	107,877	79,502	167,503	19,876
500,000	29,601	390,897	79,502	450,523	19,876
1,000,000	57,901	862,597	79,502	922,223	19,876

Maximum Loss in Each Period on Risky Investment: $3,180.

TABLE VI

EFFECT OF IMPATIENCE RATE ON OPTIMAL ALLOCATION OF CAPITAL (AT EACH DECISION POINT) WHEN $u(c) = -c^{-2}$, y = \$10,000 IN EACH PERIOD, AND r, β_2, β_3, β_4 ARE AS IN (3-1)

Capital Position x	Optimal Allocation of Capital								
	α = .80			α = .90			α = .98		
	Con-sumption	Lending (Borrowing)	Productive Investments	Con-sumption	Lending (Borrowing)	Productive Investments	Con-sumption	Lending (Borrowing)	Productive Investments
$ -200,000	-	$ (200,000)	-	-	$ (200,000)	-	-	$ (200,000)	-
-100,000	$ 11,800	(168,248)	$ 56,448	$ 8,300	(166,988)	$ 58,688	$ 5,700	(166,052)	$ 60,352
- 50,000	17,700	(152,372)	84,672	12,450	(150,482)	88,032	8,550	(149,078)	90,528
- 20,000	21,240	(142,846)	101,606	14,940	(140,578)	105,638	10,260	(138,894)	108,634
- 10,000	22,420	(139,671)	107,251	15,770	(137,277)	111,507	10,830	(135,499)	114,669
0	23,600	(136,496)	112,896	16,600	(133,976)	117,376	11,400	(132,104)	120,704
10,000	24,780	(133,321)	118,541	17,430	(130,675)	123,245	11,970	(128,709)	126,739
20,000	25,960	(130,146)	124,186	18,260	(127,374)	129,114	12,540	(125,314)	132,774
50,000	29,500	(120,620)	141,120	20,750	(117,470)	146,720	14,250	(115,130)	150,880
100,000	35,400	(104,744)	169,344	24,900	(100,964)	176,064	17,100	(98,156)	181,056
200,000	47,200	(72,992)	225,792	33,200	(67,952)	234,752	22,800	(64,208)	241,408
500,000	82,600	22,264	395,136	58,100	31,084	410,816	39,900	37,636	422,464
1,000,000	141,600	181,024	677,376	99,600	196,144	704,256	68,400	207,376	724,224
Mean Growth Rate of x + Y	-2.9%			.3%			3.9%		

TABLE VII

EFFECT OF SIZE OF (CONSTANT) NON-CAPITAL INCOME STREAM ON OPTIMAL ALLOCATION OF CAPITAL (AT EACH DECISION POINT) WHEN u(c) = log c, α = .90, AND r, β_2, β_3, β_4 ARE AS IN (3-1)

Capital Position x	Optimal Allocation of Capital								
	y = 0			y = $10,000			y = $25,000		
	Con-sumption	Lending (Borrowing)	Productive Investments	Con-sumption	Lending (Borrowing)	Productive Investments	Con-sumption	Lending (Borrowing)	Productive Investments
$ -500,000	-	-	-	-	-	-	-	$ (500,000)	-
-300,000	-	-	-	-	-	-	$ 20,000	(658,400)	$ 338,400
-200,000	-	-	-	-	$ (200,000)	-	30,000	(737,600)	507,600
-100,000	-	-	-	$ 10,000	(279,200)	$ 169,200	40,000	(816,800)	676,800
- 50,000	-	-	-	15,000	(318,800)	253,800	45,000	(856,400)	761,400
- 20,000	-	-	-	18,000	(342,560)	304,560	48,000	(880,160)	812,160
- 10,000	-	-	-	19,000	(350,480)	321,480	49,000	(888,080)	829,080
0	-	-	-	20,000	(358,400)	338,400	50,000	(896,000)	846,000
10,000	$ 1,000	$ (7,920)	$ 16,920	21,000	(366,320)	355,320	51,000	(903,920)	862,920
20,000	2,000	(15,840)	33,840	22,000	(374,240)	372,240	52,000	(911,840)	879,840
50,000	5,000	(39,600)	84,600	25,000	(398,000)	423,000	55,000	(935,600)	930,600
100,000	10,000	(79,200)	169,200	30,000	(437,600)	507,600	60,000	(975,200)	1,015,200
200,000	20,000	(158,400)	338,400	40,000	(516,800)	676,800	70,000	(1,054,400)	1,184,400
500,000	50,000	(396,000)	846,000	70,000	(754,000)	1,184,000	100,000	(1,292,000)	1,692,000
1,000,000	100,000	(792,000)	1,692,000	120,000	(1,150,400)	2,030,400	150,000	(1,688,000)	2,538,000

for a given one-period utility function, by differences in impatience, wealth, and non-capital income.

3.2 THE BALANCED MUTUAL FUND

In 2.11 we discussed the implications of our model with respect to the theory of the firm. One type of firm for which our assumptions concerning the investment opportunities hold quite well is the mutual fund. This is so because the opportunities considered by the mutual fund tend to be liquid, to be highly divisible, to have constant returns to scale, and to have proportional conversion costs.

The chief purpose of the so-called balanced mutual fund is to invest in such a (balanced) way that its share-holders should not have to make any risky investments except the purchase of shares in the fund itself.[1] Now it follows from Corollary 5 that any balanced mutual fund that wishes to sell its shares on this basis even though all individuals are in different economic circumstances (i.e., their wealths and non-capital income streams are different), possess varying degrees of impatience, and behave so as to maximize expected utility from consumption over time may still have a sizable market. This market consists of all individuals whose one-period utility functions are identical, having one of the forms $u(c) = c^{\gamma}$, $0 < \gamma < 1$, $u(c) = -c^{-\gamma}$, $\gamma > 0$, $u(c) = \log c$, $u(c) = -e^{-\gamma c}$, $\gamma > 0$, assuming that U is of the form (1-5) in the first place, and who have the same probability beliefs with respect to the returns from risky investments. In view of our conjecture at the beginning of 2.11, it would be

[1] See, for example, Douglas Hayes, Investments: Analysis and Management, New York, MacMillan, 1961, p. 398.

surprising if <u>any</u> utility functions not included in the preceding statement exist for which a given balanced mutual fund would have appeal to rational (in the sense of the expected utility principle) individuals in non-identical economic circumstances.

A sensible objective for the balanced mutual fund desiring to attract a large number of investors in a world of rational but heterogeneous individuals, then, is that given in Theorem 6. Even if the possibility of differences (among funds) in the subjective probability distributions of return were ignored, there is, even within the confines of this theorem, a theoretical basis for the existence of an infinite number of different balanced mutual funds, each characterizable by a unique risk aversion index q or q^* (as given by (2-54) and (2-93)).

3.3 ENDOWED EDUCATIONAL AND CHARITABLE ORGANIZATIONS

The model developed in this study also appears to be applicable in the case of endowed educational institutions such as private universities. All utility of such organizations appears to be derived via operating expenditures, which, if the rental value of physical assets owned and utilized are included in this category, therefore correspond to consumption in the case of an individual. The non-capital income stream consists in this case of such items as tuition and other fees, grants, and bequests. Furthermore, both financial and productive opportunities exist, and are usually taken advantage of, with respect to that part of the capital (endowment) not allocated to the current operating budget. Thus, the model developed in this paper appears to apply also to the basic decision problem facing the endowed educational organization.

Similarly, the model would be applicable to endowed charitable organizations. Here, utility is derived via donations which therefore correspond to consumption in the case of the individual. In fact, the additivity property of (1-5) appears almost harmless in this case since we would not expect much of a ratchet effect, for example, with respect to the giving of a private foundation. The non-capital income stream would be represented by new contributions to the endowment, if any. Finally, unspent funds must be invested; consequently the decision problem faced by the endowed foundation is of the same type as that which has been modeled in the case of the individual.

CHAPTER IV

RELATION TO OTHER MODELS

In this chapter, we shall examine the more important models belonging to the growing literature on normative consumption and investment behaviour. In so doing, we shall compare these models with Models I-IV and attempt to pinpoint salient similarities and differences.

4.1 FISHER'S MODEL OF THE INDIVIDUAL

As was pointed out earlier, Models I-IV are conceptually closer to Fisher's model of the individual than to any other. They may in fact be viewed as a formalization of Fisher's ideas for the class of utility functions (1-5) when risk is present. To bring the precise relationship into clearer focus, let us briefly review Fisher's solution for the two-period case.

In Fig. 2 the horizontal axis represents consumption in period 1 (c_1) and the vertical axis consumption in period 2 (c_2). The utility function $U(c_1, c_2)$ is represented by indifference curves which are convex to the origin (by the monotonicity and strict concavity of U). The individual's objective is to maximize utility over time, i.e., to reach the highest indifference curve within the limits of his resources (capital x_1 and non-capital income y_1 received at the end of the first period).

Without engaging in any transactions, the highest utility the individual can obtain is U_1 which is the utility of the consumption program (x_1, y_1) (point A). However, by borrowing in the first period, he may achieve any consumption program (c_1, c_2) lying on the line AC. Similarly, by saving in the first period he may attain any consumption

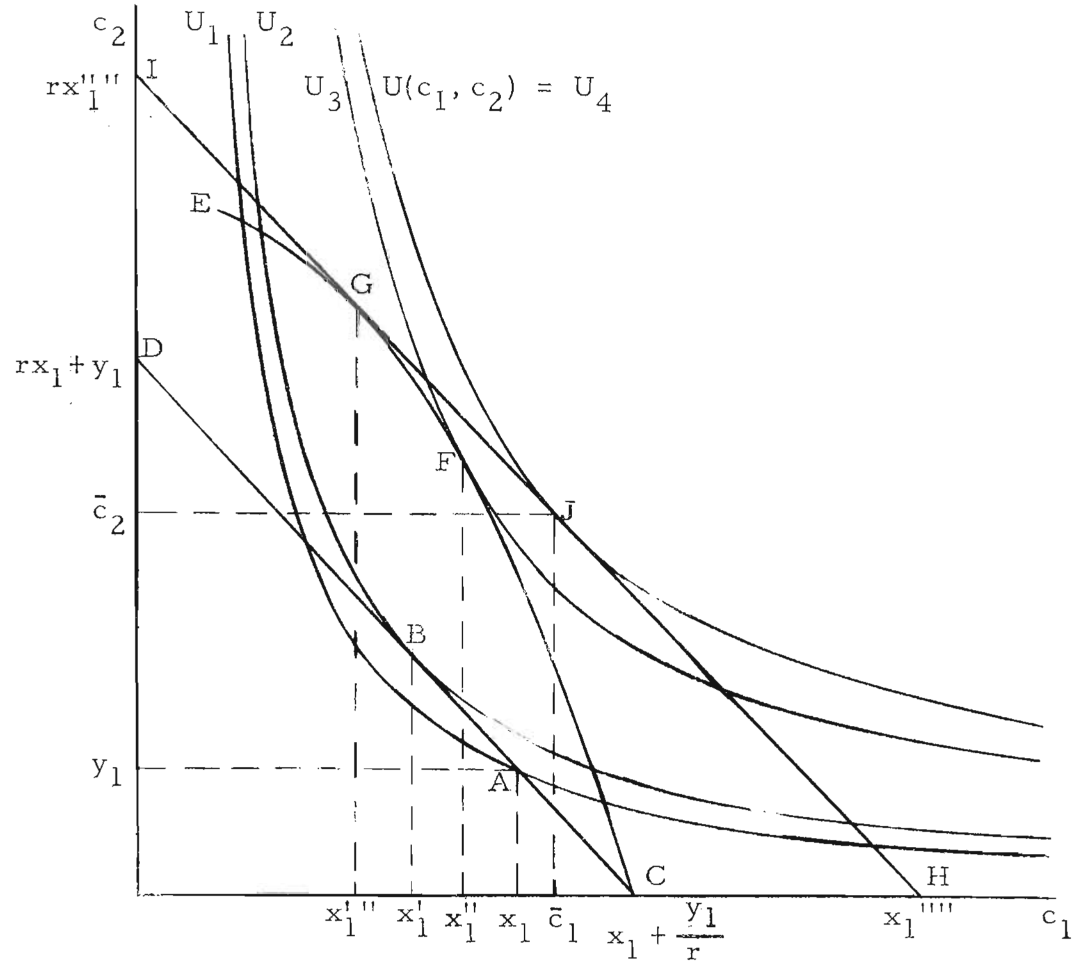

Figure 2. Fisher's Solution

combination lying on the line AD. Since U_2 is the highest utility associated with any point on CD, the individual's optimal strategy with respect to the first period, given only financial opportunities, is to consume x_1' and to save $x_1 - x_1'$. Note also that the maximum amount the individual may borrow on the security of his non-capital income is y_1/r, or the present value of his non-capital income stream (see 1.3.4, 2.8, and Theorem 5).

The locus of productive opportunities attainable by the individual (by borrowing y_1/r) is given by the curve CE. These opportunities have been ranked in descending order on the basis of their certain rate of return. By investing $x_1 + y_1/r - x_1''$ in these opportunities, the individual may attain (the maximum) utility level U_3 (point F). However, the opportunities offered him have still not been exhausted. By moving to point G on the productive opportunity curve, the individual may, by borrowing $\bar{c}_1 - x_1'''$ on the security of his productive investments, attain utility level $U_4 > U_3$ at point J on the (highest attainable) financial opportunity line HI. Thus, the individual's optimal strategy with respect to the first period is:

Borrow $y_1/r + (\bar{c}_1 - x_1''')$

Invest $(x_1 + y_1/r) - x_1'''$ (in productive opportunities)

Consume $\bar{c}_1$

This will leave $\bar{c}_2$ to be consumed in the second period.

The differences between Fisher's model and Models I-IV are now seen to be the following:

1. While Fisher's model permits the utility function to be of any meaningful form, Models I-IV are restricted to utility functions of the form (1-5) where u(c) is given by (2-15), (2-16), (2-92), or (2-45).
2. In Fisher's model, the productive investment returns are known with certainty, while in Models I-IV the returns from productive opportunities are risky.
3. In Fisher's model, the productive opportunities have diminishing returns to scale beyond some point, while in Models I-IV returns to scale are constant. Note that constant returns to scale in Fisher's model when $\beta_i > r$ for at least one i gives an infinite solution.

4. In Fisher's model, a solution is generally difficult to obtain when the horizon is more than 3 periods away since it is usually not available in analytic form. In Models I-IV, an analytic solution is available when the horizon is arbitrarily distant.

4.2 CONSUMPTION MODELS

In this section we shall examine the models which have been designed to determine how much an individual should optimally consume. Since in these investigations the only alternative to consumption generally is saving, these models are also known as savings models. We shall arbitrarily refer to those models in which the returns from savings (investments) are known in advance with certainty as the classical consumption models.

4.2.1 Classical Models

The classical consumption models may be characterized as follows:

1. The objective is to maximize a functional $H(u(c(t)))$ where $c(t) \geq 0$ is the rate of consumption at time t and u is the utility indicator of this rate. The function u is generally assumed to be strictly concave. The time horizon may be either finite or infinite. Impatience is generally present in the form of a function $\alpha(t)$.
2. The individual's resources consist of an initial capital position and may include a non-capital income received at the rate $y(t)$.
3. All assets are assumed to be invested at all times in (generally) a single opportunity. The return from this opportunity, $r(t)$, may be a function of time t but is always known with certainty. Returns to scale are usually constant.

4. Borrowing is sometimes permitted at the rate r(t) - 1.

The first classical savings model is that of Ramsey.[1] Ramsey's utility indicator had an upper bound (corresponding to bliss) but no impatience factor. While he permitted non-constant returns to scale, he did not allow borrowing. Ramsey's analysis was later extended by Samuelson and Solow to the case of several opportunities (commodities).[2] Tinbergen has also examined the classical consumption problem but without the assumption of finite bliss.[3,4]

The classical consumption model has recently been extended by Yaari. In one paper, Yaari considers the case when the individual has a bequest motive, which is represented by a utility function defined on the possible levels of wealth at the (finite) horizon point.[5] A second paper introduces two additional features: the horizon is treated as a random variable and the individual may invest in actuarial notes as an alternative to saving.[6] An actuarial note is a contract which pays a higher rate of interest (also certain) during the individual's lifetime,

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1 Frank Ramsey, "A Mathematical Theory of Saving," Economic Journal, December 1928.

2 Paul Samuelson and Robert Solow, "A Complete Capital Model Involving Heterogeneous Capital Goods," Quarterly Journal of Economics, November 1956.

3 Jan Tinbergen, "The Optimum Rate of Savings," Economic Journal, December 1956.

4 Jan Tinbergen, "Maximization of Utility over Time," Econometrica, April 1960.

5 Menahem Yaari, "On the Consumer's Lifetime Allocation Process," International Economic Review, September 1964.

6 Menahem Yaari, "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer," The Review of Economic Studies, No. 2, 1965.

but which becomes worthless upon his death. Thus, the latter model also introduces the question of portfolio selection into the individual's decision problem.

It is clear at this point that the basic difference between the classical consumption models and Models I-IV is that the returns from the available investment opportunities are assumed to be certain in the former models and risky in the latter models, except for borrowing and lending.

4.2.2 Phelps' Model

Phelps was apparently the first to introduce risk into the capital transformation of the classical consumption model.[1] He considered all capital left after the allotment to consumption to be subject to the same probability law, thus ruling out the necessity of allocating one's resources among several opportunities. This probability law was also assumed to be invariant over time. The utility function considered by Phelps was that given by (1-5). The individual was assumed to have a constant non-capital income stream but to have no recourse to borrowing. As indicated in 2.2, Phelps problem may therefore be stated as

$$\text{(2-6b)} \qquad f(x) = \max_{0 \le c \le x} \left\{ u(c) + \alpha E[f(\beta(x-c)+y)] \right\}$$

where c is bounded from above by the individual's wealth, since borrowing is not permitted.

Phelps solves (2-6b) for the utility functions (2-15) and (2-16) ($u(c) = c^{\gamma}$, $0 < \gamma < 1$, $u(c) = -c^{-\gamma}$, $\gamma > 0$), and for $u(c) = \log c$ when $y = 0$. Unfortunately, his solutions are incorrect when $y > 0$

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[1] Phelps, op. cit.

and the distribution of β is non-degenerate. For example, when $u(c) = -c^{-\gamma}$, the solution is asserted to be, letting $\bar{\beta} \equiv E[\beta^{-\gamma}]$

$$(4\text{-}1) \qquad f(x) = -\left[\frac{(\alpha\bar{\beta})^{-1/(\gamma+1)}}{(\alpha\bar{\beta})^{-1/(\gamma+1)} - 1}\right]^{\gamma+1}\left[x + \frac{y}{\bar{\beta}^{-1/\gamma} - 1}\right]^{-\gamma}$$

$$(4\text{-}2) \qquad c(x) = \text{Min}\left\{\left[1 - (\alpha\bar{\beta})^{1/(\gamma+1)}\right]\left[x + \frac{y}{\bar{\beta}^{-1/\gamma} - 1}\right], \; x\right\}$$

whenever $\alpha\bar{\beta} < 1$. But for this to be a solution, it would be necessary to be able to write

$$(4\text{-}2a) \qquad E[(\beta(x - c) + y)^{-\gamma}] = E[\beta^{-\gamma}]\left[x - c + \frac{y}{E[\beta^{-\gamma}]^{-1/\gamma}}\right]^{-\gamma}$$

which clearly is impossible unless the distribution of β is degenerate.[1] While (4-2) is correct when the distribution of β is degenerate (i. e., under certainty) for all $x \geq 0$, (4-1) holds even then only when $\alpha\beta \geq 1$ and $x \geq \left[(\alpha\bar{\beta})^{-1/(\gamma+1)} - 1\right] y/(\bar{\beta} - 1)$, i. e., when the first quantity on the right-hand side of (4-2) is less than or equal to x in all future periods.

It appears that an analytic solution to (2-6b) does not exist when the distribution of β is non-degenerate. It is ironic, therefore, that when one generalizes Phelps' problem by introducing the possibility of choice among risky investment opportunities and the opportunity to borrow and lend (see 2-6a), an analytic solution does exist. It is the second of these generalizations which guarantees the solution in closed

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[1] The right side of (4-2a) may, of course, be regarded as a first-order approximation of the left side when the variance of β is small.

form. In summary, then, Models I-IV may be viewed as a generalization of Phelps' model, to which this study owes a great debt, to the case when the individual faces any number of risky (productive) opportunities (which may depend on time), along with financial opportunities, and the opportunity to receive a not necessarily constant non-capital income stream.

4.3 INVESTMENT MODELS

In the past fifteen years, increasing attention has been focused on the subject of normative investment behaviour in the presence of risk or uncertainty. All of the investment models known to the writer which have been constructed to deal with this problem take capital as the fundamental object of choice. This is of course a significant drawback in terms of the Fisherian approach in which only alternative consumption programs are ultimately relevant for investment decisions (see 1.1). It was shown in 2.2 that when this approach is used, the utility of capital depends not only on the individual's consumption preferences but on his non-capital income, the interest rate, and the available investment opportunities and their riskiness, both present and future. The utility of capital, therefore, is truly an <u>induced</u> utility requiring for its derivation, as was seen in Chapter 2, complex logical operations which most individuals undoubtedly would find difficult to carry out in their heads. More importantly, (2-6) points out that the utility of capital is <u>not necessarily independent</u> of the other inputs used in the decision model (e.g., the return distributions). The chief criticism that one may advance against present investment models, then, is not that they assume a given utility of capital but that they do not state the conditions under which this utility function makes sense, if at all, in terms of more primitive preferences. A second drawback is that these models almost universally ignore the non-capital income, when present, of the individual.

Most normative investment models dealing with risk can be classified into one of three categories: those based on the mean-variance approach, those employing chance constraints, and those primarily concerned with long-run results.

4.3.1 The Mean-Variance Approach

The basic characteristic of the models belonging to the mean-variance category is that they are 1) nonsequential and 2) concerned only with the first and second moments of the capital position distribution at the end of the (current) period. The limitations inherent in the first of these characteristics were indicated in 1.1 and will therefore not be discussed here. The assumptions concerning investment opportunities generally coincide with those given in 1.3.3 and 1.3.4.

Objective function. In the mean-variance models, letting $\bar{x}$ (which is exogenously determined) represent the capital to be invested at a given decision point, the objective function commonly takes the form

$$(4\text{-}3) \qquad \text{Max } \{aE[X] - \text{Var}[X]\}$$

subject to

$$(4\text{-}4) \qquad \sum_{i=2}^{M} z_i \leq \bar{x} \quad \text{(unless borrowing is permitted)}$$

where

$$(4\text{-}5) \qquad X \equiv \sum_{i=2}^{M} (\beta_i - r)z_i + r\tilde{x}$$

a is a positive constant, and Var[X] is defined as the variance of the random variable X.

A modification of (4-3) is given by

$$(4\text{-}6) \qquad \text{Min Var}[X]$$

subject to (4-4) and

$$(4\text{-}7) \qquad E[X] \geq E_0 \qquad\qquad (E_0 \text{ constant})$$

A third version of (4-3) is

$$(4\text{-}8) \qquad \text{Max } E[X]$$

subject to (4-4) and

$$(4\text{-}9) \qquad \text{Var}[X] \leq V_0 \qquad\qquad (V_0 \text{ constant})$$

while a fourth modification is given by

$$(4\text{-}10) \qquad \text{Max } \frac{E[X] - d}{\{\text{Var}[X]\}^{1/2}}$$

subject to (4-4) where d is a constant. The objective function (4-3) has also been used in models in which the z_i may assume only discrete values, e. g., in capital budgeting applications.[1]

<u>Rationale behind the model.</u> There are at least four origins to which the rationale behind the mean-variance approach can be traced. (These origins are <u>not</u> in one-to-one correspondence with (4-3), (4-6),

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[1] See Joel Cord, "A Method for Allocating Funds to Investment Projects when Returns are Subject to Uncertainty," <u>Management Science</u>, January 1964.

(4-8), and (4-10).) One may be termed the intuitive explanation and goes back to Markowitz with whose name the mean-variance approach is usually associated. This explanation essentially states that "high return" is desirable and that "uncertain return" is undesirable.[1] "Return" is then defined as expected capital at the end of the period ($E[X]$) and "uncertain return" as the variance of the capital position at the end of the period ($Var[X]$). Among those who have followed Markowitz' lead and modified or extended his approach are Cheng,[2] Martin,[3] Sharpe,[4] and Baumol.[5]

A second derivation of the mean-variance criterion is the following: Let $\bar{u}(x)$ be the utility of money where $\bar{u}(x)$ is twice differentiable, strictly increasing, and strictly concave. Then, expanding $\bar{u}(X)$ into a Taylor series about the point $E(X)$, we obtain, upon dropping all terms beyond the third and translating the origin

$$(4\text{-}11) \qquad E[\bar{u}(X)] = E[X] + \frac{1}{2}\bar{u}''(E[X])\, Var[X]$$

where $\frac{1}{2}\bar{u}''(E[X]) < 0$ by the strict concavity of $\bar{u}$. This derivation is

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[1] Harry Markowitz, "Portfolio Selection," Journal of Finance, March 1952; Harry Markowitz, Portfolio Selection, New York, John Wiley, 1959, p. 6.

[2] Pao-Lun Cheng, "Optimum Bond Portfolio Selection," Management Science, July 1962.

[3] A. D. Martin, Jr., "Mathematical Programming of Portfolio Selection," Management Science, January 1954.

[4] William Sharpe, "A Simplified Model for Portfolio Analysis," Management Science, January 1963.

[5] William Baumol, "An Expected Gain-Confidence Limit Criterion for Portfolio Selection," Management Science, October 1963.

due to Marschak[1] and was later taken up by Farrar.[2]

The third starting point is that of Freund.[3] He considers the case when the utility of capital $\tilde{u}(x) = -e^{-\gamma x}(\gamma > 0)$ (which is the function (2-45)), and X is normally distributed. It is then easily shown that

$$E[\tilde{u}(X)] = E[X] - \frac{\gamma}{2} \operatorname{Var}[X]$$

which is the same function as (4-11) except for the constant.

The fourth derivation is due to Roy who proposed that investors adopt the principle "safety first". According to Roy, this principle calls for maximizing the probability that X exceed some value $d(< E[X])$.[4] By Chebyshev's inequality,

$$\Pr\{|X - E[X]| \geq E[X] - d\} \leq \frac{\operatorname{Var}[X]}{(E[X] - d)^2}$$

from which we obtain

$$\Pr\{X \leq d\} \leq \frac{\operatorname{Var}[X]}{(E[X] - d)^2}$$

If instead of minimizing the left side of the inequality we operate on the right side, we obtain (4-10).

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1 Jacob Marschak, "Why 'Should' Businessmen Maximize 'Moral Expectation'"?, Second Berkeley Symposium on Mathematical Statistics and Probability, Berkeley, University of California Press, 1951, p. 498.

2 Donald Farrar, The Investment Decision Under Uncertainty, New York, Prentice-Hall, 1962, pp. 20-21.

3 Rudolf Freund, "The Introduction of Risk Into a Programming Model", Econometrica, July 1956, p. 255.

4 A. D. Roy, "Safety First and the Holding of Assets," Econometrica, July 1952.

The mean-variance approach and the expected utility principle. Tobin[1] and Borch[2] have shown that when no restriction is placed on the distribution of X, the assumption that utility is increasing in $E[X]$ and decreasing in $Var[X]$ is consistent with the von Neumann-Morgenstern postulates only if the utility of capital is given by $\bar{u}(x) = bx - x^2$ (b constant).[3] However, if the probability distributions of return are completely specified by the first two moments, as in the case of the normal distribution, any concave utility function satisfies the consistency requirements. These limitations constitute a serious drawback to the mean-variance approach indeed as we shall demonstrate in a moment.

The separation theorem. Tobin was apparently the first to show that the optimal mix of risky investments under the mean-variance objective is independent of the total amount invested $\bar{x}$.[4] This result is usually referred to in the literature as the separation theorem and has subsequently been extended by Lintner.[5] The theorem is also readily proved by Lemma 1 and holds, as was seen in Chapter II, for utility functions other than those implied by the mean-variance criterion.

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1 James Tobin, "Liquidity Preference as Behaviour Toward Risk," The Review of Economic Studies, February 1958.

2 Karl Borch, "A Note on Utility and Attitudes to Risk," Management Science, July 1963.

3 $\bar{u}(x)$ is usually written

$$\bar{u}(x) = x - ax^2 \qquad a > 0$$

which, as Professor Marschak has kindly pointed out to me, involves a slight loss of generality.

4 Op. cit., pp. 82-84.

5 John Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," Review of Economics and Statistics, February 1965.

Limitations. Since the function $\bar{u}(x) = bx - x^2$ decreases in x for $x > b/2$, it does not have the monotonicity property that we would expect the utility of money to have. As a result, the mean-variance criterion may lead to nonsense decisions. In this conjunction, Massé has given the conditions under which an investor using the mean-variance criterion would choose a portfolio which is dominated by another feasible portfolio. The portfolio whose return is represented by the distribution function $F_1(x)$ is said to be dominated by the portfolio whose return is given by the distribution function $F_2(x)$ whenever $F_1(x) \geq F_2(x)$ for all x and $F_1(x) > F_2(x)$ for $x \epsilon P$ where $Pr\{x \epsilon P\} > 0$.[1] In this situation, we would clearly expect no individual to choose $F_1(x)$ over $F_2(x)$.

A second drawback of the quadratic utility function is that it implies increasing risk aversion, that is, $q(x) \equiv - \bar{u}''(x)/\bar{u}'(x)$ is increasing in x. As Arrow has pointed out, this implies that the total amount allocated to risky investments decreases with wealth which seems highly unrealistic.[2] Despite its current popularity, it is clear that the mean-variance approach has significant theoretical shortcomings as a prescriptive model of investment behaviour.

In summary, then, Models I-IV are considerably more general in their scope and in their approach to the investment problem than the normative models based on the mean-variance approach: the consumption program rather than capital is taken as the fundamental object of

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1 Pierre Massé, Optimal Investment Decisions, New York, Prentice-Hall, 1962, pp. 212-213.

2 Kenneth Arrow, "Comment on Duesenberry's 'The Portfolio Approach to the Demand for Money'," Review of Economics and Statistics, Supplement, February 1963.

choice, the sequential nature of the investment process is recognized explicitly instead of not at all, the decision concerning <u>how much</u> to invest and/or to borrow is endogenous rather than exogenous, and the individual's non-capital income stream is an integral part of the model instead of being outside it. Furthermore, Models I-IV possess none of the drawbacks of the mean-variance models such as increasing risk aversion and decreasing utility of capital beyond a certain capital level.

4. 3. 2 <u>Chance-Constrained Models</u>

Probably the most well-known investment model based on the method of chance constraints is that of Naslund and Whinston.[1] They consider the case of an individual with a known non-capital income stream who has <u>already decided</u> how much to spend on consumption up to a specified decision point n. The objective is postulated to be the maximization of the sum of the expected gains from investments in each period up to the horizon (decision point n) subject to two constraints. The first constraint places an upper bound on the probability that a capital loss in a given period may exceed a prescribed limit. The second constraint requires invested capital in each period to remain, with a given (minimum) probability, below a limit determined by the capital gains accumulated so far and the accumulated net savings resulting from the non-capital income after consumption requirements have been satisfied. In our notation the problem may be stated as

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[1] Bertil Naslund and Andrew Whinston, "A Model of Multi-Period Investment Under Uncertainty," <u>Management Science</u>, January 1962.

$$\text{Max } E\left[\sum_{j=1}^{n} \sum_{i=2}^{M_j} (\beta_{ij} - 1) z_{ij}\right]$$

subject to

$$\Pr\left\{\sum_{i=2}^{M_j} (\beta_{ij} - 1) z_{ij} \geq L_j\right\} \geq \xi_j \qquad j = 1, \ldots, n$$

$$\Pr\left\{\sum_{i=2}^{M_k} z_{ik} \leq x_1 - c_1 + \sum_{j=1}^{k-1}\left[\sum_{i=2}^{M_j} (\beta_{ij} - 1) z_{ij} + y_j - c_{j+1}\right]\right\} \geq \eta_k \qquad k = 1, \ldots, n$$

Since borrowing and lending are not considered by Naslund and Whinston, the financial opportunities ($i = 1$) have been left out in the representation above. A model similar to the one described has also been developed by Hillier.[1]

The basic difference between the chance-constrained models and Models I-IV is that the former leave the determination of how much should be consumed and what risks to accept outside the formal model. Thus, tradeoffs in these variables can only be evaluated informally even though these considerations are no more subjective than for example the probability distributions of return which <u>are</u> part of the model. In Models I-IV, on the other hand, these tradeoffs are automatically evaluated since a utility function (of consumption) is present which appraises all possible (ultimate) outcomes of all possible decisions. Thus the relation between the chance-constrained investment

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[1] Frederick Hillier, "The Evaluation of Risky Interrelated Investments," Technical Report No. 73, Stanford, California, Department of Statistics, Stanford University, July 1964, pp. 48-55.

models of Naslund and Whinston and of Hillier and Models I-IV is analogous to that between classical and modern decision theory.[1]

4.3.3 Long-Run Investment Models

Long-run investment models essentially fall into two classes: those which strive to make the long-run capital position as favorable as possible and those which strive to maximize the probability of surviving infinitely.

Models which maximize the long-run capital position. Consider the case in which it is desired to maximize $E[\bar{u}(x_j) | x_1]$ at some future decision point j when $x_{j+1} = \sum_{i=2}^{M} (\beta_i - r)z_{ij} + rx_j$, $j = 2, \ldots,$ and $x_1 > 0$, that is, all returns are made available for reinvestment in the next period. It has then been shown that when $\bar{u}(x_j)$ has the form (2-15), (2-16), or (2-92), the optimal investment strategies are of the form $z_{ij}(x) = p_i(\{\beta_i\}, r)x$ for all j where p_i is a proportion. In other words, the optimal investment strategy calls for investing a proportion of current capital in each opportunity, the proportions being dependent only upon the distribution function of returns.[2] As may be seen from (2-35), (2-39), and (2-43), this property of the optimal investment strategies is not lost when the objective is the maximization of utility from

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[1] For a comparison between classical statistics and modern decision theory, see, for example, Duncan Luce and Howard Raiffa, op. cit., pp. 318-324.

[2] Thomas Ferguson, "Betting Systems Which Minimize the Probability of Ruin," Journal of the Society of Industrial and Applied Mathematics, September 1965, p. 799.

consumption over time (given that the one-period utility function of consumption is of the form (2-15), (2-16), or (2-92)) and the individual receives a non-capital income stream. Analogously, we find that when the utility function $\tilde{u}(x) = -e^{-\gamma x}$ $(\gamma > 0)$, the property of nondependence on x which characterizes the optimal strategy in the long-run investment model[1] also holds in Model IV.

The particular model calling for the maximization of $E[\log x_j/x_1]$ has been considered by several authors including Kelley,[2] Breiman,[3] Bellman,[4] and Latané.[5] The optimal strategy in this case calls for maximizing the expected logarithm of capital (at the end of the period) in each period. Breiman also found that this policy is asymptotically optimal when the objective is to minimize the expected time to reach a fixed level of resources.[6] Moreover, this strategy turns out to be asymptotically optimal when the objective is to maximize the expected growth of capital.[7] As was noted in 2.7, an individual who obeys Model III (where u(c) = log c) will also behave so as to maximize the expected growth rate of the capital (remaining after the allotment to current

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1 Ibid., p. 800.

2 J. L. Kelley, "A New Interpretation of Information Rate, Bell System Technical Journal, 35, 1956.

3 Leo Breiman, "Optimal Gambling Systems for Favorable Games," Fourth Berkeley Symposium on Probability and Mathematical Statistics, Berkeley, University of California Press, 1961.

4 Richard Bellman, Adaptive Control Processes, Princeton, Princeton University Press, 1961, pp. 222-229.

5 Henry Latané, "Criteria for Choice Among Risky Ventures," Journal of Political Economy, April 1959, and Henry Latané, "Investment Criteria: A Three Asset Portfolio Balance Model," Review of Economics and Statistics, November 1963.

6 Breiman, "Optimal Gambling Systems . . . "

7 See Breiman, "Optimal Gambling Systems . . ."; Breiman, "Investment Policies . . . ,"; and Brown, op. cit.

consumption) plus the capitalized value of his (certain) non-capital income stream.

Survival models. Ferguson[1] and Truelove[2] have examined the problem of optimal investment behaviour when there is a fixed cost-of-living charge (consumption level) but no borrowing and no non-capital income stream. They postulate the objective of the individual to be the maximization of the probability of surviving infinitely, that is, of being always able to pay the cost-of-living charge.

In these models, then, the consumption level is fixed and exogeneous. However, there is clearly a tradeoff possibility between the level of consumption and the survival probability since the latter is a function of the former and it clearly makes a difference how you live, i. e., how much you consume while you survive. By introducing a utility function defined for all consumption levels and making the level of consumption a decision variable, Models I-IV, while not concerned with survival explicitly, nevertheless have the notion of survival built in. The notion of survival is in fact implicit in the utility function (of consumption) itself. If we associate survival with positive consumption in each period, it was shown in 2. 7 that unless the individual starts out in the trapping state (in which case he would perish immediately), he will survive infinitely with probability 1 in Models II and III while the survival probability may be less than 1 in Model I. Since these

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[1] Op. cit.

[2] A. J. Truelove, "A Multistage Stochastic Investment Process," RM-4025-PR, Santa Monica, The RAND Corporation, March 1964.

implications are directly related to the lower bound of the one-period utility function of consumption (see Corollary 1), we find that individuals whose utility functions have no lower bound in fact place a premium on survival.

4.3.4 Other Investment Models

All models discussed so far share the characteristic that investment decisions are made at specified, discrete points in time. However, models have also been constructed in which investment opportunities arrive randomly in time. Unless immediately accepted, each such opportunity is considered lost forever. The problem then becomes to find optimal decision rules for accepting and rejecting opportunities, which are generally viewed as long-term in nature, so as to have funds available for highly favorable opportunities which have not yet appeared while at the same time taking advantage of as many opportunities as possible. This problem has been examined by Fisher[1] and by Kaufman.[2] While consumption and borrowing and lending are not considered, Fisher does consider the case when the investor receives a non-capital income stream.[3]

Since no meaningful basis for comparing these models with Models I-IV seems to exist, we shall not review these models further. However, it appears that Models I-IV may well generalize to the random arrival situation for the case in which the investment opportunities are

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[1] James Fisher, "A Class of Stochastic Investment Problems," Operations Research, January-February 1961.

[2] Gordon Kaufman, "Sequential Investment Analysis Under Uncertainty," Journal of Business, January 1963.

[3] James Fisher, op. cit.

governed by a Markov process in which the prevailing state is not necessarily known at the time of decision.

4.4 THE STATE-PREFERENCE APPROACH: A BRIEF COMMENT

This study would not be complete without at least a brief mention of the state-preference approach to decision-making under uncertainty. This approach takes cognition of the fact that preferences may depend upon which of several possible states of the world obtains at a given time in the future. It therefore represents an important step toward a more realistic theory of intertemporal decision since the utility function would now be time-state-dependent rather than just time-dependent as in the case of Models I-IV, for example. The pioneering work in this area is that of Arrow[1]; important contributions have also been made by Hirshleifer.[2,3,4] No direct application of this approach has been made to the problem addressed in this study.

4.5 SUMMARY

The principal characteristics of the different classes of normative consumption and investment models discussed in this study have been summarized in Table VIII. Since the headings of the various entries

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[1] See Kenneth Arrow, "The Role of Securities in the Optimal Allocation of Risk-Bearing," Review of Economic Studies, April 1964.

[2] Jack Hirshleifer, "Efficient Allocation of Capital in an Uncertain World," The American Economic Review, May 1964.

[3] Jack Hirshleifer, "Investment Decision Under Uncertainty: Choice-Theoretic Approaches," The Quarterly Journal of Economics, November 1965.

[4] Jack Hirshleifer, "Investment Decision Under Uncertainty: Applications of the State-Preference Approach," to appear in The Quarterly Journal of Economics.

are self-explanatory, no further comments beyond what has already been said appear warranted.

TABLE VIII

NORMATIVE INVESTMENT AND CONSUMPTION MODELS: A COMPARATIVE SUMMARY

Model Type	Concern of Model		Is Model Sequential?	Inputs															Algorithmic or Analytic Solution
				Consumption					Utility of Capital			Productive Opportunities				Financial Opportunities	Wealth	Non-Capital Income Stream	
	Investment Decisions	Consumption Decisions			Predetermined	Constant	Strictly Concave, Monotone Utility Function	Variable Discount		Strictly Concave	Monotone Increasing		More Than One	Risky	Non-constant Returns To Scale				
Fisher 4.1	~~Yes~~	Yes	Yes	Yes	No	-	Yes	Yes	NA	-	-	Yes	Yes	No	Yes	Yes	Yes	Yes	No
Classical 4.2.1	Yes[1]	Yes	Yes	Yes	Yes	-	Yes	No	NA	-	-	Yes[2]	Yes[2]	No	Yes[3]	Yes[2]	Yes	Yes	Yes
Phelps 4.2.2	No	Yes	Yes	Yes	No	-	Yes	No	Induced	Induced	Induced	Yes	No	Yes	No	No	Yes	Yes No	No Yes
Mean-Variance 4.3.1	Yes	No	No	No	-	-	-	-	Yes	Yes	No	Yes	Yes	Yes	No	Yes	No	No	Yes
Chance-Constr. 4.3.2	Yes	No	Yes	Yes	Yes	No	-	-	No	-	-	Yes	Yes	Yes	No	Yes	Yes	Yes	Yes
Long-Run 4.3.3	Yes	No	Yes	No	-	-	-	-	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes	Yes	No	Yes
Survival 4.3.3	Yes	No	Yes	Yes	Yes	Yes	-	-	NA	-	-	Yes	No	Yes	No	No	Yes	No	Yes
Models I-IV Ch. II	Yes	Yes	Yes	Yes	No	-	Yes	No	Induced	Induced	Induced	Yes	Yes	Yes	No	Yes	Yes	Yes	Yes

NA - Not applicable

1 - Except in models of Samuelson and Solow and of Yaari, this decision is obtained as a residual

2 - Only in models of Samuelson and Solow and of Yaari does total number of opportunities exceed 1

3 - Only in models of Ramsey and of Samuelson and Solow.

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3. Maurice Allais, "L'Extension des Thèories de l'Equilibre Economique Général et du Rendement Social au Cas du Risque," Econometrica, April 1953.

4. Kenneth Arrow, Bernoulli Utility Indicators for Distributions Over Arbitrary Spaces, Technical Report No. 59, Department of Economics, Stanford University, July 1958.

5. Kenneth Arrow, "Comment on Duesenberry's 'The Portfolio Approach to the Demand for Money and Other Assets' ", Review of Economics and Statistics, Supplement, February 1963.

6. Kenneth Arrow, "The Role of Securities in the Optimal Allocation of Risk-bearing", Review of Economic Studies, April 1964.

7. William Baumol, "An Expected Gain-Confidence Limit Criterion for Portfolio Selection," Management Science, October 1963.

8. Richard Bellman, Adaptive Control Processes, Princeton, Princeton University Press, 1961.

9. Richard Bellman, Dynamic Programming, Princeton, Princeton University Press, 1957.

10. Daniel Bernoulli, "Exposition of a New Theory on the Measurement of Risk", transl. Louise Sommer, Econometrica, January 1954.

11. Harold Bierman and Seymour Smidt, The Capital Budgeting Decision, New York, MacMillan, 1960.

12. J. Black, "Optimum Savings Reconsidered, or Ramsey Without Tears", Economic Journal, June 1962.

13. Eugen von Böhm-Bawerk, Positive Theory of Capital, transl. William Smart, New York, G. E. Stechert, 1923.

14. Karl Borch, "A Note on Utility and Attitudes to Risk", Management Science, July 1963.

15. Karl Borch "A Utility Function Derived from a Survival Game," Working Paper No. 66, Western Management Science Institute, University of California, Los Angeles, February 1965.

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